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2009 J. Phys. A: Math. Theor. 42 254005

(http://iopscience.iop.org/1751-8121/42/25/254005)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 254005 (44pp) doi:10.1088/1751-8113/42/25/254005

Nonperturbative scales in AdS/CFT

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Received 2 February 2009
Published 9 June 2009
Online at stacks.iop.org/JPhysA/42/254005

Abstract

The cusp anomalous dimension is a ubiquitous quantity in four-dimensional gauge theories, ranging from QCD to maximally supersymmetric $\mathcal{N} = 4$ Yang–Mills theory, and it is one of the most thoroughly investigated observables in the AdS/CFT correspondence. In planar $\mathcal{N} = 4$ SYM theory, its perturbative expansion at weak coupling has a finite radius of convergence while at strong coupling it admits an expansion in inverse powers of the 't Hooft coupling which is given by a non-Borel summable asymptotic series. We study the cusp anomalous dimension in the transition regime from strong to weak coupling and argue that the transition is driven by nonperturbative, exponentially suppressed corrections. To compute these corrections, we revisit the calculation of the cusp anomalous dimension in planar $\mathcal{N} = 4$ SYM theory and extend the previous analysis by taking into account nonperturbative effects. We demonstrate that the scale parameterizing nonperturbative corrections coincides with the mass gap of the two-dimensional bosonic O(6) sigma model embedded into the $AdS_5 \times S^5$ string theory. This result is in agreement with the prediction coming from the string theory consideration.

PACS numbers: 11.15.Me, 11.25.Tq, 11.30.Pb

Contents

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1. Introduction

The AdS/CFT correspondence provides a powerful framework for studying maximally supersymmetric $\mathcal{N} = 4$ Yang–Mills theory (SYM) at strong coupling [1]. At present, one of the best-studied examples of the conjectured gauge/string duality is the relationship between the anomalous dimensions of Wilson operators in planar $\mathcal{N} = 4$ theory in the so-called $SL(2)$ sector and energy spectrum of folded strings spinning on $\mathrm{AdS}_5 \times \mathrm{S}^5$ [2, 3]. The Wilson operators in this sector are given by single trace operators built from L copies of the same complex scalar field and N light-cone components of the covariant derivatives. These quantum numbers define, correspondingly, the twist and the Lorentz spin of the Wilson operators in $\mathcal{N} = 4$ SYM theory (for a review, see [4]). In a dual string theory description [2, 3], they are identified as angular momenta of the string spinning on S^5 and AdS_5 parts of the background.

In general, anomalous dimensions in planar $\mathcal{N} = 4$ theory in the $SL(2)$ sector are nontrivial functions of the 't Hooft coupling $g^2 = g^2_{\mathrm{YM}} N_c/(4\pi)^2$ and quantum numbers of Wilson operators—twist L and Lorentz spin N. Significant simplification occurs in the limit [5] when the Lorentz spin grows exponentially with the twist, $L \sim \ln N$ with $N \to \infty$. In this limit, the anomalous dimensions scale logarithmically with N for arbitrary coupling and the minimal anomalous dimension has the following scaling behavior [5–9]:

$$\gamma_{N,L}(g) = [2\Gamma_{\mathrm{cusp}}(g) + \epsilon(g, j)] \ln N + \cdots, \qquad (1.1)$$

where $j = L/\ln N$ is an appropriate scaling variable and the ellipses denote terms suppressed by powers of $1/L$. Here, the coefficient in front of $\ln N$ is split into the sum of two functions in such a way that $\epsilon(g, j)$ carries the dependence on the twist and it vanishes for $j = 0$. The first term inside the square brackets in (1.1) has a universal, twist independent form [10, 11]. It involves the function of the coupling constant known as the cusp anomalous dimension. This anomalous dimension was introduced in [10] to describe specific (cusp) ultraviolet (UV) divergences of Wilson loops [12, 13] with a light-like cusp on the integration contour [14].

The cusp anomalous dimension plays a distinguished rôle in $\mathcal{N}=4$ theory and, in general, in four-dimensional Yang–Mills theories since, aside from logarithmic scaling of the anomalous dimension (1.1), it also controls infrared divergences of scattering amplitudes [15], Sudakov asymptotics of elastic form factors [16], gluon Regge trajectories [17], etc.

According to (1.1), the asymptotic behavior of the minimal anomalous dimension is determined by two independent functions, $\Gamma_{\text{cusp}}(g)$ and $\epsilon(g,j)$. At weak coupling, these functions are given by series in powers of g^2 and the first few terms of the expansion can be computed in perturbation theory. At strong coupling, the AdS/CFT correspondence allows us to obtain the expansion of $\Gamma_{\text{cusp}}(g)$ and $\epsilon(g,j)$ in powers of $1/g$ from the semiclassical expansion of the energy of the folded spinning string. Being combined together, the weak and strong coupling expansions define asymptotic behavior of these functions at the boundaries of (semi-infinite) interval $0\leqslant g<\infty$. The following questions arise: what are the corresponding interpolating functions for arbitrary g? How does the transition from the weak to strong coupling regimes occur? These are the questions that we address in this paper.

At weak coupling, the functions $\Gamma_{\text{cusp}}(g)$ and $\epsilon(g,j)$ can be found in a generic (supersymmetric) Yang–Mills theory in the planar limit by making use of the remarkable property of integrability. The Bethe ansatz approach to computing these functions at weak coupling was developed in [5, 11, 18]. It was extended in [7, 19] to all loops in $\mathcal{N}=4$ SYM theory leading to integral BES/FRS equations for $\Gamma_{\text{cusp}}(g)$ and $\epsilon(g,j)$ valid in the planar limit for arbitrary values of the scaling parameter j and the coupling constant g. For the cusp anomalous dimension, the solution to the BES equation at weak coupling is in agreement with the most advanced explicit four-loop perturbative calculation [20] and it yields a perturbative series for $\Gamma_{\text{cusp}}(g)$ which has a finite radius of convergence [19]. The BES equation was also analyzed at strong coupling [21–24] but constructing its solution for $\Gamma_{\text{cusp}}(g)$ turned out to be a nontrivial task.

The problem was solved in [25, 26], where the cusp anomalous dimension was found in the form of an asymptotic series in $1/g$. It turned out that the coefficients of this expansion have the same sign and grow factorially at higher orders. As a result, the asymptotic $1/g$ expansion of $\Gamma_{\text{cusp}}(g)$ is given by a non-Borel summable series which suffers from ambiguities that are exponentially small for $g\to\infty$. This suggests that the cusp anomalous dimension receives *nonperturbative corrections* at strong coupling [25]:

$$\Gamma_{\text{cusp}}(g)=\sum_{k=-1}^{\infty}c_k/g^k-\frac{\sigma}{4\sqrt{2}}m_{\text{cusp}}^2+o\big(m_{\text{cusp}}^2\big). \tag{1.2}$$

Here the dependence of the nonperturbative scale m_{cusp}^2 on the coupling constant $m_{\text{cusp}}\sim g^{1/4}\,\mathrm{e}^{-\pi g}$ follows, through a standard analysis [27, 28], from the large order behavior of the expansion coefficients, $c_k\sim\Gamma\big(k+\frac{1}{2}\big)$ for $k\to\infty$. The value of the coefficient σ in (1.2) depends on the regularization of Borel singularities in the perturbative $1/g$ expansion, and the numerical prefactor was introduced for later convenience.

Note that the expression for the nonperturbative scale m_{cusp}^2 looks similar to that for the mass gap in an asymptotically free field theory with the coupling constant $\sim 1/g$. An important difference is, however, that m_{cusp}^2 is a dimensionless function of the 't Hooft coupling. This is perfectly consistent with the fact that the $\mathcal{N}=4$ model is a conformal field theory and, therefore, it does not involve any dimensionfull scale. Nevertheless, as we will show in this paper, the nonperturbative scale m_{cusp}^2 is indeed related to the mass gap in the *two-dimensional* bosonic O(6) sigma model.

Relation (1.2) sheds light on the properties of $\Gamma_{\text{cusp}}(g)$ in the transition region $g\sim 1$. Going from $g\gg 1$ to $g=1$, we find that m_{cusp}^2 increases and, as a consequence, nonperturbative $O\left(m_{\rm cusp}^2\right)$ corrections to $\Gamma_{\rm cusp}(g)$ become comparable with perturbative $O(1/g)$ corrections. We will argue in this paper that the nonperturbative corrections play a crucial role in the transition from the strong to weak coupling regime. To describe the transition, we present a simplified model for the cusp anomalous dimension. This model correctly captures the properties of $\Gamma_{\rm cusp}(g)$ at strong coupling and, most importantly, it allows us to obtain a closed expression for the cusp anomalous dimension which turns out to be remarkably close to the exact value of $\Gamma_{\rm cusp}(g)$ throughout the entire range of the coupling constant.

In the AdS/CFT correspondence, relation (1.2) should follow from the semiclassical expansion of the energy of the quantized folded spinning string [2, 3]. On the right-hand side of (1.2), the coefficient c_{-1} corresponds to the classical energy and c_k describes $(k+1)$th loop correction. Indeed, the explicit two-loop stringy calculation [29] yields the expressions for c_{-1}, c_0 and c_1 which are in a perfect agreement with (1.2)[4]. However, the semiclassical approach does not allow us to calculate nonperturbative corrections to $\Gamma_{\rm cusp}(g)$, and verification of (1.2) remains a challenge for the string theory.

Recently, Alday and Maldacena [6] put forward an interesting proposal that the scaling function $\epsilon(g, j)$ entering (1.1) can be found exactly at strong coupling in terms of a nonlinear O(6) bosonic sigma model embedded into the $\mathrm{AdS}_5 \times \mathrm{S}^5$ model. More precisely, using the dual description of Wilson operators as folded strings spinning on $\mathrm{AdS}_5 \times \mathrm{S}^5$ and taking into account the one-loop stringy corrections to these states [8], they conjectured that the scaling function $\epsilon(g, j)$ should be related at strong coupling to the energy density $\epsilon_{\rm O(6)}$ in the ground state of the O(6) model corresponding to the particle density $\rho_{\rm O(6)} = j/2$:

$$\epsilon_{\rm O(6)} = \frac{\epsilon(g, j) + j}{2}, \qquad m_{\rm O(6)} = k g^{1/4}\,\mathrm{e}^{-\pi g}\,[1 + O(1/g)]\,. \tag{1.3}$$

This relation should hold at strong coupling and $j/m_{\rm O(6)} =$ fixed. Here, the scale $m_{\rm O(6)}$ is identified as the dynamically generated mass gap in the O(6) model with $k = 2^{3/4}\pi^{1/4}\big/\Gamma\left(\frac{5}{4}\right)$ being the normalization factor.

The O(6) sigma model is an exactly solvable theory [31–34], and the dependence of $\epsilon_{\rm O(6)}$ on the mass scale $m_{\rm O(6)}$ and the density of particles $\rho_{\rm O(6)}$ can be found exactly with the help of thermodynamical Bethe ansatz equations. Together with (1.3), this allows us to determine the scaling function $\epsilon(g, j)$ at strong coupling. In particular, for $j/m_{\rm O(6)} \ll 1$, the asymptotic behavior of $\epsilon(g, j)$ follows from the known expression for the energy density of the O(6) model in the (nonperturbative) regime of small density of particles [6, 34–36]:

$$\epsilon(j, g) + j = m^2\left[\frac{j}{m} + \frac{\pi^2}{24}\left(\frac{j}{m}\right)^3 + O(j^4/m^4)\right], \tag{1.4}$$

with $m \equiv m_{\rm O(6)}$. For $j/m_{\rm O(6)} \gg 1$, the scaling function $\epsilon(g, j)$ admits a perturbative expansion in inverse powers of g with the coefficients enhanced by powers of $\ln\ell$ (with $\ell = j/(4g) \ll 1$ [6, 8].

$$\epsilon(g, j) + j = 2\ell^2\left[g + \frac{1}{\pi}\left(\frac{3}{4} - \ln\ell\right) + \frac{1}{4\pi^2 g}\left(\frac{q_{02}}{2} - 3\ln\ell + 4(\ln\ell)^2\right) + \mathcal{O}(1/g^2)\right] + O(\ell^4).$$

This expansion was derived both in string theory [37] and in gauge theory [30, 38, 39] yielding, however, different results for the constant q_{02}. The reason for the disagreement remains unclear. The first two terms of this expansion were also computed in string theory [37] and they were found to be in a disagreement with gauge theory calculation [30, 38, 39].

[4] The same result was obtained using a different approach from the quantum string Bethe ansatz in [9, 30].

Remarkably enough, relation (1.3) was established in planar $\mathcal{N} = 4$ SYM theory at strong coupling [35] using the conjectured integrability of the dilatation operator [7]. The mass scale $m_{O(6)}$ was computed both numerically [36] and analytically [35, 38], and it was found to be in a perfect agreement with (1.3). This result is extremely nontrivial given the fact that the scale $m_{O(6)}$ has a different origin in gauge and in string theory sides of AdS/CFT. In string theory, it is generated by the dimensional transmutation mechanism in *two-dimensional* effective theory describing dynamics of massless modes in the $AdS_5 \times S^5$ sigma model. In gauge theory, the same scale parameterizes nonperturbative corrections to anomalous dimensions in *four-dimensional* Yang–Mills theory at strong coupling. It is interesting to note that a similar phenomenon, when two different quantities computed in four-dimensional gauge theory and in dual two-dimensional sigma model coincide, has already been observed in the BPS spectrum in $\mathcal{N} = 2$ supersymmetric Yang–Mills theory [40, 41]. We would like the mention that the precise matching of the leading coefficients in perturbative expansion of spinning string energy and anomalous dimensions on the gauge side was previously found in [42, 43, 44]. Relation (1.3) implies that for the anomalous dimensions (1.1) the gauge/string correspondence holds at the level of nonperturbative corrections.

As we just explained, the functions $\Gamma_{\rm cusp}(g)$ and $\epsilon(g, j)$ entering (1.1) receive nonperturbative contributions at strong coupling described by the scales $m_{\rm cusp}$ and $m_{O(6)}$, respectively. In $\mathcal{N} = 4$ SYM theory, these functions satisfy two different integral equations [7, 19] and there is no *a priori* reason why the scales $m_{\rm cusp}$ and $m_{O(6)}$ should be related to each other. Nevertheless, examining their leading order expressions, equations (1.2) and (1.3), we note that they have the same dependence on the coupling constant. One may wonder whether subleading $O(1/g)$ corrections are also related to each other. In this paper, we show that the two scales coincide at strong coupling to any order of $1/g$ expansion:

$$m_{\rm cusp} = m_{O(6)}, \tag{1.5}$$

thus proving that nonperturbative corrections to the cusp anomalous dimension (1.2) and to the scaling function (1.4) are parameterized by the same scale.

Relations (1.2) and (1.5) also have an interpretation in string theory. The cusp anomalous dimension has the meaning of the energy density of a folded string spinning on AdS_3 [2, 6]. As such, it receives quantum corrections from both massive and massless excitations of this string in the $AdS_5 \times S^5$ sigma model. The O(6) model emerges in this context as the effective theory describing the dynamics of massless modes. In distinction with the scaling function $\epsilon(g, j)$, for which the massive modes decouple in the limit $j/m_{O(6)}$ = fixed and $g \to \infty$, the cusp anomalous dimension is not described entirely by the O(6) model. Nevertheless, it is expected that the leading nonperturbative corrections to $\Gamma_{\rm cusp}(g)$ should originate from nontrivial infrared dynamics of the massless excitations and, therefore, they should be related to nonperturbative corrections to the vacuum energy density in the O(6) model. As a consequence, $\Gamma_{\rm cusp}(g)$ should receive exponentially suppressed corrections proportional to the square of the O(6) mass gap $\sim m_{O(6)}^2$. We show in this paper by explicit calculation that this is indeed the case.

The paper is organized as follows. In section 2, we revisit the calculation of the cusp anomalous dimension in planar $\mathcal{N} = 4$ SYM theory and construct the exact solution for $\Gamma_{\rm cusp}(g)$. In section 3, we analyze the expressions obtained at strong coupling and identify nonperturbative corrections to $\Gamma_{\rm cusp}(g)$. In section 4, we compute subleading corrections to the nonperturbative scales $m_{\rm cusp}$ and $m_{O(6)}$ and show that they are the same for the two scales. Then, we extend our analysis to higher orders in $1/g$ and demonstrate that the two scales coincide. Section 5 contains concluding remarks. Some technical details of our calculations are presented in appendices.

2. Cusp anomalous dimension in $\mathcal{N} = 4$ SYM

The cusp anomalous dimension can be found in planar $\mathcal{N} = 4$ SYM theory for arbitrary coupling as a solution to the BES equation [19]. At strong coupling, $\Gamma_{\text{cusp}}(g)$ was constructed in [25, 26] in the form of perturbative expansion in $1/g$. The coefficients of this series grow factorially at higher orders, thus indicating that $\Gamma_{\text{cusp}}(g)$ receives nonperturbative corrections which are exponentially small at strong coupling (equation (1.2)). To identity such corrections, we revisit in this section the calculation of the cusp anomalous dimension and construct the exact solution to the BES equation for arbitrary coupling.

2.1. Integral equation and mass scale

In the Bethe ansatz approach, the cusp anomalous dimension is determined by the behavior around the origin of the auxiliary function $\gamma(t)$ related to the density of Bethe roots:

$$\Gamma_{\text{cusp}}(g) = -8\mathrm{i}g^2 \lim_{t\to 0} \gamma(t)/t. \tag{2.1}$$

The function $\gamma(t)$ depends on the 't Hooft coupling and has the form

$$\gamma(t) = \gamma_+(t) + \mathrm{i}\gamma_-(t), \tag{2.2}$$

where $\gamma_\pm(t)$ are real functions of t with a definite parity $\gamma_\pm(\pm t) = \pm\gamma_\pm(t)$. For arbitrary coupling, the functions $\gamma_\pm(t)$ satisfy the (infinite-dimensional) system of integral equations:

$$\begin{aligned} &\int_0^\infty \frac{\mathrm{d}t}{t} J_{2n-1}(t)\left[\frac{\gamma_-(t)}{1-\mathrm{e}^{-t/(2g)}} + \frac{\gamma_+(t)}{\mathrm{e}^{t/(2g)}-1}\right] = \frac{1}{2}\delta_{n,1}, \\ &\int_0^\infty \frac{\mathrm{d}t}{t} J_{2n}(t)\left[\frac{\gamma_+(t)}{1-\mathrm{e}^{-t/(2g)}} - \frac{\gamma_-(t)}{\mathrm{e}^{t/(2g)}-1}\right] = 0, \end{aligned} \tag{2.3}$$

with $n \geqslant 1$ and $J_n(t)$ being the Bessel functions. These relations are equivalent to the BES equation [19] provided that $\gamma_\pm(t)$ verify certain analyticity conditions specified in section 2.2.

As was shown in [25, 35], equations (2.3) can be significantly simplified with the help of the transformation $\gamma(t) \to \Gamma(t)$[5]:

$$\Gamma(t) = \left(1 + \mathrm{i}\coth\frac{t}{4g}\right)\gamma(t) \equiv \Gamma_+(t) + \mathrm{i}\Gamma_-(t). \tag{2.4}$$

We find from (2.1) and (2.4) the following representation for the cusp anomalous dimension:

$$\Gamma_{\text{cusp}}(g) = -2g\Gamma(0). \tag{2.5}$$

It follows from (2.2) and (2.3) that $\Gamma_\pm(t)$ are real functions with a definite parity, $\Gamma_\pm(-t) = \pm\Gamma_\pm(t)$, satisfying the system of integral equations:

$$\begin{aligned} &\int_0^\infty \mathrm{d}t\, \cos(ut)[\Gamma_-(t) - \Gamma_+(t)] = 2, \\ &\int_0^\infty \mathrm{d}t\, \sin(ut)[\Gamma_-(t) + \Gamma_+(t)] = 0, \end{aligned} \tag{2.6}$$

with u being an arbitrary real parameter such that $-1 \leqslant u \leqslant 1$. Since $\Gamma_\pm(t)$ take real values, we can rewrite these relations in a compact form:

$$\int_0^\infty \mathrm{d}t[\mathrm{e}^{\mathrm{i}ut}\Gamma_-(t) - \mathrm{e}^{-\mathrm{i}ut}\Gamma_+(t)] = 2. \tag{2.7}$$

[5] With a slight abuse of notations, we use here the same notation as for the Euler gamma function.

To recover (2.3), we apply (2.4), replace in (2.6) the trigonometric functions by their Bessel series expansions

$$\cos(ut) = 2\sum_{n\geqslant 1}(2n-1)\frac{\cos((2n-1)\varphi)}{\cos\varphi}\frac{J_{2n-1}(t)}{t},$$
$$\sin(ut) = 2\sum_{n\geqslant 1}(2n)\frac{\sin(2n\varphi)}{\cos\varphi}\frac{J_{2n}(t)}{t}, \tag{2.8}$$

with $u = \sin\varphi$, and finally compare coefficients in front of $\cos((2n-1)\varphi)/\cos\varphi$ and $\sin(2n\varphi)/\cos\varphi$ on both sides of (2.6). It is important to stress that, doing this calculation, we interchanged the sum over n with the integral over t. This is only justified for φ real and, therefore, relation (2.6) only holds for $-1 \leqslant u \leqslant 1$.

Comparing (2.7) and (2.3), we observe that the transformation $\gamma_\pm \to \Gamma_\pm$ eliminates the dependence of the integral kernel on the left-hand side of (2.7) on the coupling constant. One may then wonder where does the dependence of the functions $\Gamma_\pm(t)$ on the coupling constant come from? We will show in the following subsection that it is dictated by additional conditions imposed on analytical properties of solutions to (2.7).

Relations (2.5) and (2.6) were used in [25] to derive an asymptotic (perturbative) expansion of $\Gamma_{\text{cusp}}(g)$ in powers of $1/g$. This series however suffers from Borel singularities, and we expect that the cusp anomalous dimension should receive nonperturbative corrections $\sim \mathrm{e}^{-2\pi g}$ exponentially small at strong coupling. As was already mentioned in section 1, similar corrections are also present in the scaling function $\epsilon(g, j)$ which controls the asymptotic behavior of the anomalous dimensions (1.1) in the limit when the Lorentz spin of Wilson operators grows exponentially with their twist. According to (1.3), for $j/m_{O(6)} =$ fixed and $g \to \infty$, the scaling function coincides with the energy density of the O(6) model embedded into $\mathrm{AdS}_5 \times \mathrm{S}^5$. The mass gap of this model defines a new nonperturbative scale $m_{O(6)}$ in AdS/CFT. Its dependence on the coupling g follows univocally from the FRS equation and it has the following form [35, 38]:

$$m_{O(6)} = \frac{8\sqrt{2}}{\pi^2}\,\mathrm{e}^{-\pi g} - \frac{8g}{\pi}\,\mathrm{e}^{-\pi g}\mathrm{Re}\left[\int_0^\infty \frac{\mathrm{d}t\,\mathrm{e}^{\mathrm{i}(t-\pi/4)}}{t+\mathrm{i}\pi g}(\Gamma_+(t)+\mathrm{i}\Gamma_-(t))\right], \tag{2.9}$$

where $\Gamma_\pm(t)$ are solutions to (2.7). To compute the mass gap (2.9), we have to solve the integral equation (2.7) and, then, substitute the resulting expression for $\Gamma_\pm(t)$ into (2.9). Note that the same functions also determine the cusp anomalous dimension (2.5).

Later in the paper, we will construct a solution to the integral equation (2.7) and, then, apply (2.5) to compute nonperturbative corrections to $\Gamma_{\text{cusp}}(g)$ at strong coupling.

2.2. Analyticity conditions

The integral equations (2.7) and (2.3) determine $\Gamma_\pm(t)$ and $\gamma_\pm(t)$, or equivalently the functions $\Gamma(t)$ and $\gamma(t)$, up to a contribution of zero modes. The latter satisfy the same integral equations (2.7) and (2.3) but without an inhomogeneous term on the right-hand side.

To fix the zero modes, we have to impose additional conditions on solutions to (2.7) and (2.3). These conditions follow unambiguously from the BES equation [23, 25] and they can be formulated as a requirement that $\gamma_\pm(t)$ should be entire functions of t which admit a representation in the form of Neumann series over the Bessel functions

$$\gamma_-(t) = 2\sum_{n\geqslant 1}(2n-1)J_{2n-1}(t)\gamma_{2n-1}, \qquad \gamma_+(t) = 2\sum_{n\geqslant 1}(2n)J_{2n}(t)\gamma_{2n}, \tag{2.10}$$

with the expansion coefficients γ_{2n-1} and γ_{2n} depending on the coupling constant. This implies in particular that the series on the right-hand side of (2.10) are convergent on the real axis. Using orthogonality conditions for the Bessel functions, we obtain from (2.10)

$$\gamma_{2n-1} = \int_0^\infty \frac{\mathrm{d}t}{t} J_{2n-1}(t)\gamma_-(t), \qquad \gamma_{2n} = \int_0^\infty \frac{\mathrm{d}t}{t} J_{2n}(t)\gamma_+(t). \tag{2.11}$$

Here, we assumed that the sum over n on the right-hand side of (2.10) can be interchanged with the integral over t. We will show below that relations (2.10) and (2.11) determine a unique solution to the system (2.3).

The coefficient γ_1 plays a special role in our analysis since it determines the cusp anomalous dimension (2.1):

$$\Gamma_{\mathrm{cusp}}(g) = 8g^2\gamma_1(g). \tag{2.12}$$

Here we applied (2.2) and (2.10) and took into account small-t behavior of the Bessel functions, $J_n(t) \sim t^n$ as $t \to 0$.

Let us now translate (2.10) and (2.11) into properties of the functions $\Gamma_\pm(t)$, or equivalently $\Gamma(t)$. It is convenient to rewrite relation (2.4) as

$$\Gamma(\mathrm{i}t) = \gamma(\mathrm{i}t)\frac{\sin\left(\frac{t}{4g}+\frac{\pi}{4}\right)}{\sin\left(\frac{t}{4g}\right)\sin\left(\frac{\pi}{4}\right)} = \gamma(\mathrm{i}t)\sqrt{2}\prod_{k=-\infty}^{\infty}\frac{t-4\pi g\left(k-\frac{1}{4}\right)}{t-4\pi gk}. \tag{2.13}$$

Since $\gamma(\mathrm{i}t)$ is an entire function in the complex t-plane, we conclude from (2.13) that $\Gamma(\mathrm{i}t)$ has an infinite number of zeros, $\Gamma(\mathrm{i}t_{\mathrm{zeros}}) = 0$, and poles, $\Gamma(\mathrm{i}t) \sim 1/(t - t_{\mathrm{poles}})$, on a real t-axis located at

$$t_{\mathrm{zeros}} = 4\pi g\left(\ell - \tfrac{1}{4}\right), \qquad t_{\mathrm{poles}} = 4\pi g\ell', \tag{2.14}$$

where $\ell, \ell' \in \mathbb{Z}$ and $\ell' \neq 0$ so that $\Gamma(\mathrm{i}t)$ is regular at the origin (see equation (2.1)). Note that $\Gamma(\mathrm{i}t)$ has an additional (infinite) set of zeros coming from the function $\gamma(\mathrm{i}t)$ but, in distinction with (2.14), their position is not fixed. Later in the paper, we will construct a solution to the integral equation (2.6) which satisfies relations (2.14).

2.3. Toy model

To understand the relationship between analytical properties of $\Gamma(\mathrm{i}t)$ and properties of the cusp anomalous dimension, it is instructive to slightly simplify the problem and consider a 'toy' model in which the function $\Gamma(\mathrm{i}t)$ is replaced with $\Gamma^{(\mathrm{toy})}(\mathrm{i}t)$.

We require that $\Gamma^{(\mathrm{toy})}(\mathrm{i}t)$ satisfies the same integral equation (2.6) and define, following (2.5), the cusp anomalous dimension in the toy model as

$$\Gamma^{(\mathrm{toy})}_{\mathrm{cusp}}(g) = -2g\Gamma^{(\mathrm{toy})}(0). \tag{2.15}$$

The only difference compared to $\Gamma(\mathrm{i}t)$ is that $\Gamma^{(\mathrm{toy})}(\mathrm{i}t)$ has different analytical properties dictated by the relation

$$\Gamma^{(\mathrm{toy})}(\mathrm{i}t) = \gamma^{(\mathrm{toy})}(\mathrm{i}t)\frac{t+\pi g}{t}, \tag{2.16}$$

while $\gamma^{(\mathrm{toy})}(\mathrm{i}t)$ has the same analytical properties as the function $\gamma(\mathrm{i}t)$[6]. This relation can be considered as a simplified version of (2.13). Indeed, it can be obtained from (2.13) if we

[6] Note that the function $\gamma^{(\mathrm{toy})}(t)$ does no longer satisfy the integral equation (2.3). Substitution of (2.16) into (2.7) yields an integral equation for $\gamma^{(\mathrm{toy})}(t)$ which can be obtained from (2.3) by replacing $1/(1-\mathrm{e}^{-t/(2g)}) \to \frac{\pi g}{2t} + \frac{1}{2}$ and $1/(\mathrm{e}^{t/(2g)} - 1) \to \frac{\pi g}{2t} - \frac{1}{2}$ in the kernel on the left-hand side of (2.3).

retained in the product only one term with $k = 0$. As compared with (2.14), the function $\Gamma^{(\text{toy})}(\mathrm{i}t)$ does not have poles and it vanishes for $t = -\pi g$.

The main advantage of the toy model is that, as we will show in section 2.8, the expression for $\Gamma^{(\text{toy})}_{\text{cusp}}(g)$ can be found in a closed form for an arbitrary value of the coupling constant (see equation (2.50)). We will then compare it with the exact expression for $\Gamma_{\text{cusp}}(g)$ and identify the difference between the two functions.

2.4. Exact bounds and unicity of the solution

Before we turn to finding a solution to (2.6), let us demonstrate that this integral equation, supplemented with the additional conditions (2.10) and (2.11) on its solutions, leads to nontrivial constraints for the cusp anomalous dimension valid for arbitrary coupling g.

Let us multiply both sides of the two relations in (2.3) by $2(2n-1)\gamma_{2n-1}$ and $2(2n)\gamma_{2n}$, respectively, and perform summation over $n \geqslant 1$. Then, we convert the sums into functions $\gamma_\pm(t)$ using (2.10) and add the second relation to the first one to obtain[7]

$$\gamma_1 = \int_0^\infty \frac{\mathrm{d}t}{t} \frac{(\gamma_+(t))^2 + (\gamma_-(t))^2}{1 - \mathrm{e}^{-t/(2g)}}. \tag{2.17}$$

Since $\gamma_\pm(t)$ are real functions of t and the denominator is positively definite for $0 \leqslant t < \infty$, this relation leads to the following inequality:

$$\gamma_1 \geqslant \int_0^\infty \frac{\mathrm{d}t}{t} (\gamma_-(t))^2 \geqslant 2\gamma_1^2 \geqslant 0. \tag{2.18}$$

Here, we replaced the function $\gamma_-(t)$ by its Bessel series (2.10) and made use of the orthogonality condition for the Bessel functions with odd indices. We deduce from (2.18) that

$$0 \leqslant \gamma_1 \leqslant \tfrac{1}{2} \tag{2.19}$$

and, then, apply (2.12) to translate this inequality into the following relation for the cusp anomalous dimension:

$$0 \leqslant \Gamma_{\text{cusp}}(g) \leqslant 4g^2. \tag{2.20}$$

We would like to stress that this relation should hold in planar $\mathcal{N} = 4$ SYM theory for arbitrary coupling g.

Note that the lower bound on the cusp anomalous dimension, $\Gamma_{\text{cusp}}(g) \geqslant 0$, holds in any gauge theory [11]. It is the upper bound $\Gamma_{\text{cusp}}(g) \leqslant 4g^2$ that is a distinguished feature of $\mathcal{N} = 4$ theory. Let us verify the validity of (2.20). At weak coupling, $\Gamma_{\text{cusp}}(g)$ admits perturbative expansion in powers of g^2 [20]:

$$\Gamma_{\text{cusp}}(g) = 4g^2\left[1 - \frac{1}{3}\pi^2 g^2 + \frac{11}{45}\pi^4 g^4 - 2\left(\frac{73}{630}\pi^6 + 4\zeta_3^2\right)g^6 + \cdots\right], \tag{2.21}$$

while at strong coupling it has the form [25, 26, 29]

$$\Gamma_{\text{cusp}}(g) = 2g\left[1 - \frac{3\ln 2}{4\pi}g^{-1} - \frac{\mathrm{K}}{16\pi^2}g^{-2} - \left(\frac{3\mathrm{K}\ln 2}{64\pi^3} + \frac{27\zeta_3}{2048\pi^3}\right)g^{-3} + O(g^{-4})\right], \tag{2.22}$$

with K being the Catalan constant. It is easy to see that relations (2.21) and (2.22) are in an agreement with (2.20).

For arbitrary g, we can verify relation (2.20) by using the results for the cusp anomalous dimension obtained from a numerical solution of the BES equation [25, 45]. The comparison is shown in figure 1. We observe that the upper bound condition $\Gamma_{\text{cusp}}(g)/(2g) \leqslant 2g$ is indeed satisfied for arbitrary $g > 0$.

[7] Our analysis here goes along the same lines as in appendix A of [35].

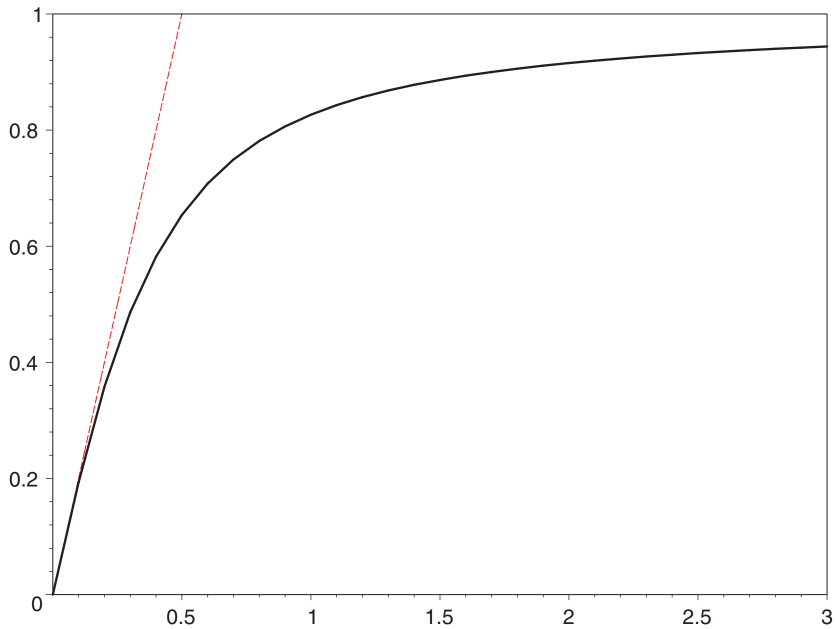

Figure 1. Dependence of the cusp anomalous dimension $\Gamma_{\text{cusp}}(g)/(2g)$ on the coupling constant. The dashed line denotes the upper bound $2g$.

We are ready to show that the analyticity conditions formulated in section 2.2 specify a unique solution to (2.3). As was already mentioned, solutions to (2.3) are defined modulo contribution of zero modes, $\gamma(t) \to \gamma(t)+\gamma^{(0)}(t)$, with $\gamma^{(0)}(t)$ being a solution to homogenous equations. Going through the same steps that led us to (2.17), we obtain

$$0 = \int_0^\infty \frac{\mathrm{d}t}{t} \frac{(\gamma_+^{(0)}(t))^2 + (\gamma_-^{(0)}(t))^2}{1 - \mathrm{e}^{-t/(2g)}}, \tag{2.23}$$

where the zero on the left-hand side is due to absence of the inhomogeneous term. Since the integrand is a positively definite function, we immediately deduce that $\gamma^{(0)}(t) = 0$ and, therefore, the solution for $\gamma(t)$ is unique.

2.5. *Riemann–Hilbert problem*

Let us now construct the exact solution to the integral equations (2.7) and (2.3). To this end, it is convenient to Fourier transform functions (2.2) and (2.4):

$$\widetilde{\Gamma}(k) = \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{2\pi}\, \mathrm{e}^{\mathrm{i}kt}\,\Gamma(t), \qquad \widetilde{\gamma}(k) = \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{2\pi}\, \mathrm{e}^{\mathrm{i}kt}\,\gamma(t). \tag{2.24}$$

According to (2.2) and (2.10), the function $\gamma(t)$ is given by the Neumann series over the Bessel functions. Then, we perform the Fourier transform on both sides of (2.10) and use the well-known fact that the Fourier transform of the Bessel function $J_n(t)$ vanishes for $k^2 > 1$ to deduce that the same is true for $\gamma(t)$ leading to

$$\widetilde{\gamma}(k) = 0, \qquad \text{for} \quad k^2 > 1. \tag{2.25}$$

This implies that the Fourier integral for $\gamma(t)$ only involves modes with $-1 \leqslant k \leqslant 1$ and, therefore, the function $\gamma(t)$ behaves at large (complex) t as

$$\gamma(t) \sim \mathrm{e}^{|t|}, \qquad \text{for} \quad |t| \to \infty. \tag{2.26}$$

Let us now examine the function $\widetilde{\Gamma}(k)$. We find from (2.24) and (2.13) that $\widetilde{\Gamma}(k)$ admits the following representation:

$$\widetilde{\Gamma}(k) = \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{2\pi}\, \mathrm{e}^{\mathrm{i}kt} \frac{\sinh\left(\frac{t}{4g} + \mathrm{i}\frac{\pi}{4}\right)}{\sinh\left(\frac{t}{4g}\right)\sin\left(\frac{\pi}{4}\right)}\gamma(t). \tag{2.27}$$

Here, the integrand has poles along the imaginary axis at $t = 4\pi\mathrm{i}gn$ (with $n = \pm 1, \pm 2, \ldots)$[8].

It is suggestive to evaluate the integral (2.27) by deforming the integration contour to infinity and by picking up residues at the poles. However, taking into account relation (2.26), we find that the contribution to (2.27) at infinity can be neglected for $k^2 > 1$ only. In this case, closing the integration contour into the upper (or lower) half-plane for $k > 1$ (or $k < -1$) we find

$$\widetilde{\Gamma}(k) \overset{k^2 \geq 1}{=} \theta(k-1)\sum_{n\geqslant 1} c_+(n,g)\,\mathrm{e}^{-4\pi ng(k-1)} + \theta(-k-1)\sum_{n\geqslant 1} c_-(n,g)\,\mathrm{e}^{-4\pi ng(-k-1)}. \tag{2.28}$$

Here, the notation was introduced for k-independent expansion coefficients:

$$c_\pm(n,g) = \mp 4g\gamma(\pm 4\pi\mathrm{i}gn)\,\mathrm{e}^{-4\pi ng}, \tag{2.29}$$

where the factor $\mathrm{e}^{-4\pi ng}$ is inserted to compensate the exponential growth of $\gamma(\pm 4\pi\mathrm{i}gn) \sim \mathrm{e}^{4\pi ng}$ at large n (see equation (2.26)). For $k^2 \leqslant 1$, we are not allowed to neglect the contribution to (2.27) at infinity and relation (2.28) does no longer hold. As we will see in a moment, for $k^2 \leqslant 1$ the function $\widetilde{\Gamma}(k)$ can be found from (2.7).

Comparing relations (2.25) and (2.28), we conclude that, in distinction with $\widetilde{\gamma}(k)$, the function $\widetilde{\Gamma}(k)$ does not vanish for $k^2 > 1$. Moreover, each term on the right-hand side of (2.28) is exponentially small at strong coupling and the function scales at large k as $\widetilde{\Gamma}(k) \sim \mathrm{e}^{-4\pi g(|k|-1)}$. This implies that nonzero values of $\widetilde{\Gamma}(k)$ for $k^2 > 1$ are of nonperturbative origin. Indeed, in a perturbative approach of [25], the function $\Gamma(t)$ is given by the Bessel function series analogous to (2.10) and, similar to (2.25), the function $\widetilde{\Gamma}(k)$ vanishes for $k^2 > 1$ to any order in $1/g$ expansion.

We note that the sum on the right-hand side of (2.28) runs over poles of the function $\Gamma(\mathrm{i}t)$ specified in (2.14). We recall that in the toy model (2.16), $\Gamma^{(\text{toy})}(\mathrm{i}t)$ and $\gamma^{(\text{toy})}(\mathrm{i}t)$ are entire functions of t. At large t they have the same asymptotic behavior as the Bessel functions, $\Gamma^{(\text{toy})}(\mathrm{i}t) \sim \gamma^{(\text{toy})}(\mathrm{i}t) \sim \mathrm{e}^{\pm it}$. Performing their Fourier transformation (2.24), we find

$$\widetilde{\gamma}^{(\text{toy})}(k) = \widetilde{\Gamma}^{(\text{toy})}(k) = 0, \qquad \text{for} \quad k^2 > 1, \tag{2.30}$$

in a close analogy with (2.25). Comparison with (2.28) shows that the coefficients (2.29) vanish in the toy model for arbitrary n and g:

$$c_+^{(\text{toy})}(n,g) = c_-^{(\text{toy})}(n,g) = 0. \tag{2.31}$$

Relation (2.28) defines the function $\widetilde{\Gamma}(k)$ for $k^2 > 1$ but it involves the coefficients $c_\pm(n,g)$ that need to be determined. In addition, we have to construct the same function for $k^2 \leqslant 1$. To achieve both goals, let us return to the integral equations (2.6) and replace $\Gamma_\pm(t)$ by Fourier integrals (see equations (2.24) and (2.4)):

$$\begin{aligned} \Gamma_+(t) &= \int_{-\infty}^{\infty} \mathrm{d}k\,\cos(kt)\widetilde{\Gamma}(k), \\ \Gamma_-(t) &= -\int_{-\infty}^{\infty} \mathrm{d}k\,\sin(kt)\widetilde{\Gamma}(k). \end{aligned} \tag{2.32}$$

[8] We recall that $\gamma(t) = O(t)$ and, therefore, the integrand is regular at $t = 0$.

In this way, we obtain from (2.6) the following remarkably simple integral equation for $\widetilde{\Gamma}(k)$:

$$\fint_{-\infty}^{\infty} \frac{\mathrm{d}k\widetilde{\Gamma}(k)}{k-u} + \pi\widetilde{\Gamma}(u) = -2, \qquad (-1 \leqslant u \leqslant 1), \tag{2.33}$$

where the integral is defined using the principal value prescription. This relation is equivalent to the functional equation obtained in [24] (see equation (55) there). Let us split the integral in (2.33) into $k^2 \leqslant 1$ and $k^2 > 1$ and rewrite (2.33) in the form of a singular integral equation for the function $\widetilde{\Gamma}(k)$ on the interval $-1 \leqslant k \leqslant 1$:

$$\widetilde{\Gamma}(u) + \frac{1}{\pi}\fint_{-1}^{1} \frac{\mathrm{d}k\widetilde{\Gamma}(k)}{k-u} = \phi(u), \qquad (-1 \leqslant u \leqslant 1), \tag{2.34}$$

where the inhomogeneous term is given by

$$\phi(u) = -\frac{1}{\pi}\left(2 + \int_{-\infty}^{-1} \frac{\mathrm{d}k\widetilde{\Gamma}(k)}{k-u} + \int_{1}^{\infty} \frac{\mathrm{d}k\widetilde{\Gamma}(k)}{k-u}\right). \tag{2.35}$$

Since integration in (2.35) goes over $k^2 > 1$, the function $\widetilde{\Gamma}(k)$ can be replaced on the right-hand side of (2.35) by its expression (2.28) in terms of the coefficients $c_\pm(n, g)$.

The integral equation (2.34) can be solved by standard methods [46]. A general solution for $\widetilde{\Gamma}(k)$ reads as (for $-1 \leqslant k \leqslant 1$)

$$\widetilde{\Gamma}(k) = \frac{1}{2}\phi(k) - \frac{1}{2\pi}\left(\frac{1+k}{1-k}\right)^{1/4} \fint_{-1}^{1} \frac{\mathrm{d}u\phi(u)}{u-k}\left(\frac{1-u}{1+u}\right)^{1/4} - \frac{\sqrt{2}}{\pi}\left(\frac{1+k}{1-k}\right)^{1/4} \frac{c}{1+k}, \tag{2.36}$$

where the last term describes the zero mode contribution with c being an arbitrary function of the coupling. We replace $\phi(u)$ by its expression (2.35), interchange the order of integration and find after some algebra

$$\widetilde{\Gamma}(k) \overset{k^2\leqslant 1}{=} -\frac{\sqrt{2}}{\pi}\left(\frac{1+k}{1-k}\right)^{1/4}\left[1 + \frac{c}{1+k} + \frac{1}{2}\int_{-\infty}^{\infty} \frac{\mathrm{d}p\widetilde{\Gamma}(p)}{p-k}\left(\frac{p-1}{p+1}\right)^{1/4}\theta(p^2-1)\right]. \tag{2.37}$$

Note that the integral on the right-hand side of (2.37) goes along the real axis except the interval $[-1, 1]$ and, therefore, $\widetilde{\Gamma}(p)$ can be replaced by its expression (2.28).

Being combined together, relations (2.28) and (2.37) define the function $\widetilde{\Gamma}(k)$ for $-\infty < k < \infty$ in terms of a (infinite) set of yet unknown coefficients $c_\pm(n, g)$ and $c(g)$. To fix these coefficients, we will first perform the Fourier transform of $\widetilde{\Gamma}(k)$ to obtain the function $\Gamma(t)$ and, then, require that $\Gamma(t)$ should have correct analytical properties (2.14).

2.6. General solution

We are now ready to write down a general expression for the function $\Gamma(t)$. According to (2.24), it is related to $\widetilde{\Gamma}(k)$ through the inverse Fourier transformation

$$\Gamma(t) = \int_{-1}^{1} \mathrm{d}k\, \mathrm{e}^{-\mathrm{i}kt}\widetilde{\Gamma}(k) + \int_{-\infty}^{-1} \mathrm{d}k\, \mathrm{e}^{-\mathrm{i}kt}\widetilde{\Gamma}(k) + \int_{1}^{\infty} \mathrm{d}k\, \mathrm{e}^{-\mathrm{i}kt}\widetilde{\Gamma}(k), \tag{2.38}$$

where we split the integral into three terms since $\widetilde{\Gamma}(k)$ has a different form for $k < -1$, $-1 \leqslant k \leqslant 1$ and $k > 1$. Then, we use the expressions obtained for $\widetilde{\Gamma}(k)$, equations (2.28) and (2.37), to find after some algebra the following remarkable relation (see appendix B for details):

$$\Gamma(\mathrm{i}t) = f_0(t)V_0(t) + f_1(t)V_1(t). \tag{2.39}$$

Here, the notation was introduced for

$$f_0(t) = -1 + \sum_{n\geqslant 1} t\left[c_+(n,g)\frac{U_1^+(4\pi ng)}{4\pi ng - t} + c_-(n,g)\frac{U_1^-(4\pi ng)}{4\pi ng + t}\right],$$
$$f_1(t) = -c(g) + \sum_{n\geqslant 1} 4\pi ng\left[c_+(n,g)\frac{U_0^+(4\pi ng)}{4\pi ng - t} + c_-(n,g)\frac{U_0^-(4\pi ng)}{4\pi ng + t}\right]. \tag{2.40}$$

Also, V_n and $U_n^{\pm}$ (with $n = 0, 1$) stand for integrals

$$V_n(x) = \frac{\sqrt{2}}{\pi}\int_{-1}^{1} \mathrm{d}u(1+u)^{1/4-n}(1-u)^{-1/4}\,\mathrm{e}^{ux},$$
$$U_n^{\pm}(x) = \frac{1}{2}\int_{1}^{\infty} \mathrm{d}u(u\pm 1)^{-1/4}(u\mp 1)^{1/4-n}\,\mathrm{e}^{-(u-1)x}, \tag{2.41}$$

which can be expressed in terms of the Whittaker functions of first and second kinds [47] (see appendix D). We would like to emphasize that solution (2.39) is exact for arbitrary coupling $g > 0$ and that the only undetermined ingredients in (2.39) are the expansion coefficients $c_{\pm}(n, g)$ and $c(g)$.

In the special case of the toy model, equation (2.31), the expansion coefficients vanish, $c_{\pm}^{(\mathrm{toy})}(n, g) = 0$, and relation (2.40) takes a simple form

$$f_0^{(\mathrm{toy})}(t) = -1, \qquad f_1^{(\mathrm{toy})}(t) = -c^{(\mathrm{toy})}(g). \tag{2.42}$$

Substituting these expressions into (2.39), we obtain a general solution to the integral equation (2.7) in the toy model:

$$\Gamma^{(\mathrm{toy})}(\mathrm{i}t) = -V_0(t) - c^{(\mathrm{toy})}(g)V_1(t). \tag{2.43}$$

It involves an arbitrary g-dependent constant $c^{(\mathrm{toy})}$ which will be determined in section 2.8.

2.7. Quantization conditions

Relation (2.39) defines a general solution to the integral equation (2.7). It still depends on the coefficients $c_{\pm}(n, g)$ and $c(g)$ that need to be determined. We recall that $\Gamma(\mathrm{i}t)$ should have poles and zeros specified in (2.14).

Let us first examine poles on the right-hand side of (2.39). It follows from (2.41) that $V_0(t)$ and $V_1(t)$ are entire functions of t and, therefore, poles can only come from the functions $f_0(t)$ and $f_1(t)$. Indeed, the sums entering (2.40) produce an infinite sequence of poles located at $t = \pm 4\pi n$ (with $n \geqslant 1$) and, as a result, solution (2.39) has a correct pole structure (2.14). Let us now require that $\Gamma(\mathrm{i}t)$ should vanish for $t = t_{\mathrm{zero}}$ specified in (2.14). This leads to an infinite set of relations

$$\Gamma\big(4\pi\mathrm{i}g\big(\ell - \tfrac{1}{4}\big)\big) = 0, \qquad \ell \in \mathbb{Z}. \tag{2.44}$$

Replacing $\Gamma(\mathrm{i}t)$ by its expression (2.39), we rewrite these relations in an equivalent form

$$f_0(t_\ell)V_0(t_\ell) + f_1(t_\ell)V_1(t_\ell) = 0, \qquad t_\ell = 4\pi g\big(\ell - \tfrac{1}{4}\big). \tag{2.45}$$

Relations (2.44) and (2.45) provide the quantization conditions for the coefficients $c(g)$ and $c_{\pm}(n, g)$, respectively, that we will analyze in section 3.

Let us substitute (2.39) into expression (2.5) for the cusp anomalous dimension. The result involves the functions $V_n(t)$ and $f_n(t)$ (with $n = 1, 2$) evaluated at $t = 0$. It is easy to see from (2.41) that $V_0(0) = 1$ and $V_1(0) = 2$. In addition, we obtain from (2.40) that $f_0(0) = -1$ for arbitrary coupling leading to

$$\Gamma_{\mathrm{cusp}}(g) = 2g[1 - 2f_1(0)]. \tag{2.46}$$

Replacing $f_1(0)$ by its expression (2.40), we find the following relation for the cusp anomalous dimension in terms of the coefficients c and $c_\pm$:

$$\Gamma_{\rm cusp}(g) = 2g\left\{1 + 2c(g) - 2\sum_{n\geqslant 1}\left[c_-(n,g)U_0^-(4\pi n g) + c_+(n,g)U_0^+(4\pi n g)\right]\right\}. \tag{2.47}$$

We would like to stress that relations (2.46) and (2.47) are exact and hold for arbitrary coupling g. This implies that, at weak coupling, it should reproduce the known expansion of $\Gamma_{\rm cusp}(g)$ in *positive integer* powers of g^2 [20]. Similarly, at strong coupling, it should reproduce the known $1/g$ expansion [25, 26] and, most importantly, describe nonperturbative, exponentially suppressed corrections to $\Gamma_{\rm cusp}(g)$.

2.8. Cusp anomalous dimension in the toy model

As before, the situation simplifies for the toy model (2.43). In this case, we have only one quantization condition $\Gamma^{(\rm toy)}(-\pi i g) = 0$ which follows from (2.16). Together with (2.43), it allows us to fix the coefficient $c^{(\rm toy)}(g)$ as

$$c^{(\rm toy)}(g) = -\frac{V_0(-\pi g)}{V_1(-\pi g)}. \tag{2.48}$$

Then, we substitute relations (2.48) and (2.31) into (2.47) and obtain

$$\Gamma^{(\rm toy)}_{\rm cusp}(g) = 2g[1 + 2c^{(\rm toy)}(g)] = 2g\left[1 - 2\frac{V_0(-\pi g)}{V_1(-\pi g)}\right]. \tag{2.49}$$

Replacing $V_0(-\pi g)$ and $V_1(-\pi g)$ by their expressions in terms of the Whittaker function of the first kind (see equation (D.2)), we find the following remarkable relation:

$$\Gamma^{(\rm toy)}_{\rm cusp}(g) = 2g\left[1 - (2\pi g)^{-1/2}\frac{M_{1/4,1/2}(2\pi g)}{M_{-1/4,0}(2\pi g)}\right], \tag{2.50}$$

which defines the cusp anomalous dimension in the toy model for arbitrary coupling $g > 0$.

Using (2.50), it is straightforward to compute $\Gamma^{(\rm toy)}_{\rm cusp}(g)$ for arbitrary positive g. By construction, $\Gamma^{(\rm toy)}_{\rm cusp}(g)$ should be different from $\Gamma_{\rm cusp}(g)$. Nevertheless, evaluating (2.50) for $0 \leqslant g \leqslant 3$, we found that the numerical values of $\Gamma^{(\rm toy)}_{\rm cusp}(g)$ are very close to the exact values of the cusp anomalous dimension shown by the solid line in figure 1. Also, as we will show in a moment, the two functions have similar properties at strong coupling. To compare these functions, it is instructive to examine the asymptotic behavior of $\Gamma^{(\rm toy)}_{\rm cusp}(g)$ at weak and strong couplings.

2.8.1. Weak coupling.

At weak coupling, we find from (2.50)

$$\Gamma^{(\rm toy)}_{\rm cusp}(g) = \frac{3}{2}\pi g^2 - \frac{1}{2}\pi^2 g^3 - \frac{1}{64}\pi^3 g^4 + \frac{5}{64}\pi^4 g^5 - \frac{11}{512}\pi^5 g^6 - \frac{3}{512}\pi^6 g^7 + O(g^8). \tag{2.51}$$

Comparison with (2.21) shows that this expansion is quite different from the weak coupling expansion of the cusp anomalous dimension. In distinction with $\Gamma_{\rm cusp}(g)$, the expansion in (2.51) runs in both even and odd powers of the coupling. In addition, the coefficient in front of g^n on the right-hand side of (2.51) has transcendentality $(n-1)$ while for $\Gamma_{\rm cusp}(g)$ it equals $(n-2)$ (with n taking even values only).

Despite this and similar to the weak coupling expansion of the cusp anomalous dimension [19], the series (2.51) has a finite radius of convergence $|g_0| = 0.796$. It is determined by the position of the zero of the Whittaker function closest to the origin, $M_{-1/4,0}(2\pi g_0) = 0$ for

$g_0 = -0.297 \pm \mathrm{i}0.739$. Moreover, numerical analysis indicates that $\Gamma^{(\text{toy})}_{\text{cusp}}(g)$ has an infinite number of poles in the complex g-plane. The poles are located on the left-half side of the complex plane, $\text{Re}g < 0$, symmetrically with respect to the real axis, and they approach progressively the imaginary axis as one goes away from the origin.

2.8.2. Strong coupling. At strong coupling, we can replace the Whittaker functions in (2.50) by their asymptotic expansion for $g \gg 1$. It is convenient however to apply (2.49) and replace the functions $V_0(-\pi g)$ and $V_1(-\pi g)$ by their expressions given in (D.14) and (D.16), respectively. In particular, we have (see equation (D.14))

$$V_0(-\pi g) = \mathrm{e}^{1/(2\alpha)} \frac{\alpha^{5/4}}{\Gamma\left(\frac{3}{4}\right)} \left[F\left(\frac{1}{4}, \frac{5}{4} | \alpha + i0\right) + \Lambda^2 F\left(-\frac{1}{4}, \frac{3}{4} | -\alpha\right)\right], \qquad \alpha = 1/(2\pi g), \tag{2.52}$$

where the parameter Λ^2 is defined as

$$\Lambda^2 = \sigma \alpha^{-1/2}\,\mathrm{e}^{-1/\alpha} \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)}, \qquad \sigma = \mathrm{e}^{-3\mathrm{i}\pi/4}. \tag{2.53}$$

Here, $F(a, b| - \alpha)$ is expressed in terms of the confluent hypergeometric function of the second kind (see equations (D.12) and (D.7) in appendix D and equation (2.56)) [47]:

$$\begin{aligned} F\left(\frac{1}{4}, \frac{5}{4} | -\alpha\right) &= \alpha^{-5/4} U_0^{+}\left(1/(2\alpha)\right) \Big/ \Gamma\left(\frac{5}{4}\right), \\ F\left(-\frac{1}{4}, \frac{3}{4} | -\alpha\right) &= \alpha^{-3/4} U_0^{-}\left(1/(2\alpha)\right) \Big/ \Gamma\left(\frac{3}{4}\right). \end{aligned} \tag{2.54}$$

The function $F(a, b| - \alpha)$ defined in this way is an analytical function of α with a cut along the negative semi-axis.

For positive $\alpha = 1/(2\pi g)$, the function $F\left(-\frac{1}{4}, \frac{3}{4}| - \alpha\right)$ entering (2.52) is defined away from the cut and its large g expansion is given by the Borel-summable asymptotic series (for $a = -\frac{1}{4}$ and $b = \frac{3}{4}$):

$$F(a, b| - \alpha) = \sum_{k\geqslant 0} \frac{(-\alpha)^k}{k!} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(a)\Gamma(b)} = 1 - \alpha ab + O(\alpha^2), \tag{2.55}$$

with the expansion coefficients growing factorially to higher orders in α. This series can be immediately resummed by means of the Borel resummation method. Namely, replacing $\Gamma(a+k)$ by its integral representation and performing the sum over k we find for $\text{Re}\alpha > 0$

$$F(a, b| - \alpha) = \frac{\alpha^{-a}}{\Gamma(a)} \int_0^\infty \mathrm{d}s\, s^{a-1}(1+s)^{-b}\,\mathrm{e}^{-s/\alpha}, \tag{2.56}$$

in agreement with (2.54) and (2.41).

Relation (2.55) holds in fact for arbitrary complex α and the functions $F(a, b|\alpha \pm \mathrm{i}0)$, defined for $\alpha > 0$ above and below the cut, respectively, are given by the same asymptotic expansion (2.55) with α replaced by $-\alpha$. The important difference is that now the series (2.55) is no longer Borel summable. Indeed, if one attempted to resum this series using the Borel summation method, one would immediately find a branch point singularity along the integration contour at $s = 1$:

$$F(a, b|\alpha \pm \mathrm{i}0) = \frac{\alpha^{-a}}{\Gamma(a)} \int_0^\infty \mathrm{d}s\, s^{a-1}(1 - s \mp \mathrm{i}0)^{-b}\,\mathrm{e}^{-s/\alpha}. \tag{2.57}$$

The ambiguity related to the choice of the prescription to integrate over the singularity is known as Borel ambiguity. In particular, deforming the s-integration contour above or below the cut, one obtains two different functions $F(a, b|\alpha \pm \mathrm{i}0)$. They define analytical continuation of the same function $F(a, b|-\alpha)$ from $\mathrm{Re}\,\alpha > 0$ to the upper and lower edges of the cut running along the negative semi-axis. Its discontinuity across the cut, $F(a, b|\alpha + \mathrm{i}0) - F(a, b|, \alpha - \mathrm{i}0)$, is exponentially suppressed at small $\alpha > 0$ and is proportional to the nonperturbative scale Λ^2 (see equation (D.17)). This property is perfectly consistent with the fact that function (2.52) is an entire function of α. Indeed, it takes the same form if one used $\alpha - \mathrm{i}0$ prescription in the first term in the right-hand side of (2.52) and replaced σ in (2.53) by its complex conjugated value.

We can now elucidate the reason for decomposing the entire V_0-function in (2.52) into the sum of two F-functions. In spite of the fact that the analytical properties of the former function are simpler compared to the latter functions, its asymptotic behavior at large g is more complicated. Indeed, the F-functions admit asymptotic expansions in the whole complex g-plane and they can be unambiguously defined through the Borel resummation once their analytical properties are specified (we recall that the function $F(a, b|\alpha)$ has a cut along the positive semi-axis). In distinction with this, the entire function $V_0(-\pi g)$ admits different asymptotic behavior for positive and negative values of g in virtue of the Stokes phenomenon. Not only does it restrict the domain of validity of each asymptotic expansion, but it also forces us to keep track of both perturbative and nonperturbative contributions in the transition region from positive to negative g, including the transition from the strong to weak coupling.

We are now in position to discuss the strong coupling expansion of the cusp anomalous dimension in the toy model, including into our consideration both perturbative and nonperturbative contributions. Substituting (2.52) and a similar relation for $V_1(-\pi g)$ (see equation (D.16)) into (2.49), we find (for $\alpha^+ \equiv \alpha + \mathrm{i}0$ and $\alpha = 1/(2\pi g)$)

$$\Gamma_{\mathrm{cusp}}^{(\mathrm{toy})}(g)/(2g) = 1 - \alpha \frac{F\left(\frac{1}{4}, \frac{5}{4}|\alpha^+\right) + \Lambda^2 F\left(-\frac{1}{4}, \frac{3}{4}|-\alpha\right)}{F\left(\frac{1}{4}, \frac{1}{4}|\alpha^+\right) + \frac{1}{4}\Lambda^2 \alpha F\left(\frac{3}{4}, \frac{3}{4}|-\alpha\right)}. \tag{2.58}$$

Since the parameter Λ^2 is exponentially suppressed at strong coupling, equation (2.53), and, at the same time, the F-functions are all of the same order, it makes sense to expand the right-hand side of (2.58) in powers of Λ^2 and, then, study separately each coefficient function. In this way, we identify the leading, Λ^2 independent term as perturbative contribution to $\Gamma_{\mathrm{cusp}}^{(\mathrm{toy})}(g)$ and the $O(\Lambda^2)$ term as the leading nonperturbative correction. More precisely, expanding the right-hand side of (2.58) in powers of Λ^2, we obtain

$$\Gamma_{\mathrm{cusp}}^{(\mathrm{toy})}(g)/(2g) = C_0(\alpha) - \alpha\Lambda^2 C_2(\alpha) + \tfrac{1}{4}\alpha^2\Lambda^4 C_4(\alpha) + O(\Lambda^6). \tag{2.59}$$

Here, the expansion runs in even powers of Λ and the coefficient functions $C_k(\alpha)$ are given by algebraic combinations of F-functions:

$$C_0 = 1 - \alpha \frac{F\left(\frac{1}{4}, \frac{5}{4}|\alpha^+\right)}{F\left(\frac{1}{4}, \frac{1}{4}|\alpha^+\right)}, \qquad C_2 = \frac{1}{\left[F\left(\frac{1}{4}, \frac{1}{4}|\alpha^+\right)\right]^2}, \qquad C_4 = \frac{F\left(\frac{3}{4}, \frac{3}{4}|-\alpha\right)}{\left[F\left(\frac{1}{4}, \frac{1}{4}|\alpha^+\right)\right]^3}, \tag{2.60}$$

where we applied (D.9) and (D.12) to simplify the last two relations. Since the coefficient functions are expressed in terms of the functions $F(a, b|\alpha^+)$ and $F(a, b|-\alpha)$ having the cut along the positive and negative semi-axes, respectively, $C_k(\alpha)$ are analytical functions of α in the upper-half plane.

Let us now examine the strong coupling expansion of the coefficient functions (2.60). Replacing F-functions in (2.60) by their asymptotic series representation (2.55), we get

$$C_0 = 1 - \alpha - \frac{1}{4}\alpha^2 - \frac{3}{8}\alpha^3 - \frac{61}{64}\alpha^4 - \frac{433}{128}\alpha^5 + O(\alpha^6),$$
$$C_2 = 1 - \frac{1}{8}\alpha - \frac{11}{128}\alpha^2 - \frac{151}{1024}\alpha^3 - \frac{13085}{32768}\alpha^4 + O(\alpha^5), \tag{2.61}$$
$$C_4 = 1 - \frac{3}{4}\alpha - \frac{27}{32}\alpha^2 - \frac{317}{128}\alpha^3 + O(\alpha^4).$$

Not surprisingly, these expressions inherit the properties of the F-functions—the series (2.61) are asymptotic and non-Borel summable. If one simply substituted relations (2.61) into the right-hand side of (2.59), one would then worry about the meaning of nonperturbative $O(\Lambda^2)$ corrections to (2.59) given the fact that the strong coupling expansion of perturbative contribution $C_0(\alpha)$ suffers from Borel ambiguity. We recall that the appearance of exponentially suppressed corrections to $\Gamma^{(\text{toy})}_{\text{cusp}}(g)$ is ultimately related to the Stokes phenomenon for the function $V_0(-\pi g)$ (equation (2.52)). As was already mentioned, this does not happen for the F-function and, as a consequence, its asymptotic expansion, supplemented with the additional analyticity conditions, allows us to reconstruct the F-function through the Borel transformation (equations (2.56) and (2.57)). Since the coefficient functions (2.60) are expressed in terms of the F-functions, we may expect that the same should be true for the C-functions. Indeed, it follows from the unicity condition of asymptotic expansion [27] that the functions $C_0(\alpha), C_2(\alpha), C_4(\alpha), \ldots$ are uniquely determined by their series representations (2.61) as soon as the latter are understood as asymptotic expansions for the functions analytical in the upper-half plane $\text{Im}\,\alpha \geqslant 0$. This implies that the exact expressions for functions (2.60) can be unambiguously constructed by means of the Borel resummation but the explicit construction remains beyond the scope of the present study.

Since expression (2.58) is exact for arbitrary coupling g, we may now address the question formulated in section 1: how does the transition from the strong to the weak coupling regime occur? We recall that, in the toy model, $\Gamma^{(\text{toy})}_{\text{cusp}}(g)/(2g)$ is given for $g \ll 1$ and $g \gg 1$ by relations (2.51) and (2.59), respectively. Let us choose some sufficiently small value of the coupling constant, say $g = 1/4$, and compute $\Gamma^{(\text{toy})}_{\text{cusp}}(g)/(2g)$ using three different representations. First, we substitute $g = 0.25$ into (2.58) and find the exact value as 0.4424(3). Then, we use the weak coupling expansion (2.51) and obtain a close value 0.4420(2). Finally, we use the strong coupling expansion (2.59) and evaluate the first few terms on the right-hand side of (2.59) for $g = 0.25$ to get

$$\begin{aligned}\text{Equation}(2.59) &= (0.2902 - 0.1434\text{i}) + (0.1517 + 0.1345\text{i}) \\ &\quad + (0.0008 + 0.0086\text{i}) - (0.0002 - 0.0003\text{i}) + \cdots = 0.4425 + \cdots.\end{aligned} \tag{2.62}$$

Here the four expressions inside the round brackets correspond to contributions proportional to $\Lambda^0, \Lambda^2, \Lambda^4$ and Λ^6, respectively, with $\Lambda^2(g = 0.25) = 0.3522 \times \text{e}^{-3\text{i}\pi/4}$ being the nonperturbative scale (2.53).

We observe that each term in (2.62) takes complex values and their sum is remarkably close to the exact value. In addition, the leading $O(\Lambda^2)$ nonperturbative correction (the second term) is comparable with the perturbative correction (the first term). Moreover, the former term starts to dominate over the latter one as we go to smaller values of the coupling constant. Thus, the transition from the strong to weak coupling regime is driven by nonperturbative corrections parameterized by the scale Λ^2. Moreover, the numerical analysis indicates that the expansion of $\Gamma^{(\text{toy})}_{\text{cusp}}(g)$ in powers of Λ^2 is convergent for $\text{Re}\,g > 0$.

2.8.3. From toy model to the exact solution. Relation (2.59) is remarkably similar to the expected strong coupling expansion of the cusp anomalous dimension (1.2) with the function $C_0(\alpha)$ providing perturbative contribution and Λ^2 defining the leading nonperturbative contribution. Let us compare $C_0(\alpha)$ with the known perturbative expansion (2.22) of $\Gamma_{\rm cusp}(g)$. In terms of the coupling $\alpha = 1/(2\pi g)$, the first few terms of this expansion look as

$$\Gamma_{\rm cusp}(g)/(2g) = 1 - \frac{3\ln 2}{2}\alpha - \frac{\rm K}{4}\alpha^2 - \left(\frac{3{\rm K}\ln 2}{8} + \frac{27\zeta_3}{256}\right)\alpha^3 + \dots, \qquad (2.63)$$

where the ellipses denote both higher order corrections in α and nonperturbative corrections in Λ^2. Comparing (2.63) and the first term, $C_0(\alpha)$, on the right-hand side of (2.59), we observe that both expressions approach the same value 1 as $\alpha \to 0$.

As was already mentioned, the expansion coefficients of the two series have different transcendentality—they are rational for the toy model (equation (2.61)) and have maximal transcendentality for the cusp anomalous dimension (equation (2.63)). Note that the two series would coincide if one formally replaced the transcendental numbers in (2.63) by appropriate rational constants. In particular, replacing

$$\frac{3\ln 2}{2} \to 1, \qquad \frac{\rm K}{2} \to \frac{1}{2}, \qquad \frac{9\zeta_3}{32} \to \frac{1}{3}, \qquad \dots, \qquad (2.64)$$

one obtains from (2.63) the first few terms of perturbative expansion (2.61) of the function C_0 in the toy model. This rule can be generalized to all loops as follows. Introducing an auxiliary parameter τ, we define the generating function for the transcendental numbers in (2.64) and rewrite (2.64) as

$$\exp\left[\frac{3\ln 2}{2}\tau - \frac{\rm K}{2}\tau^2 + \frac{9\zeta_3}{32}\tau^3 + \dots\right] \to \exp\left[\tau - \frac{\tau^2}{2} + \frac{\tau^3}{3} + \dots\right]. \qquad (2.65)$$

Going to higher loops, we have to add higher order terms in τ to both exponents. On the right-hand side, these terms are resummed into $\exp(\ln(1+\tau)) = 1+\tau$, while on the left-hand side, they produce the ratio of Euler gamma functions leading to

$$\frac{\Gamma\left(\frac14\right)\Gamma\left(1+\frac{\tau}{4}\right)\Gamma\left(\frac34-\frac{\tau}{4}\right)}{\Gamma\left(\frac34\right)\Gamma\left(1-\frac{\tau}{4}\right)\Gamma\left(\frac14+\frac{\tau}{4}\right)} \to (1+\tau). \qquad (2.66)$$

Taking logarithms on both sides of this relation and subsequently expanding them in powers of τ, we obtain the substitution rules which generalize (2.64) to the complete family of transcendental numbers entering into the strong coupling expansion (2.63). At this point, relation (2.66) can be thought of as an empirical rule, which allows us to map the strong coupling expansion of the cusp anomalous dimension (2.63) into that in the toy model (equation (2.61)). We will clarify its origin in section 4.2.

In spite of the fact that the numbers entering both sides of (2.64) have different transcendentality, we may compare their numerical values. Taking into account that $3\ln 2/2 = 1.0397(2)$, ${\rm K}/2 = 0.4579(8)$ and $9\zeta_3/32 = 0.3380(7)$, we observe that relation (2.64) defines a meaningful approximation to the transcendental numbers. Moreover, examining the coefficients in front of τ^n on both sides of (2.65) at large n, we find that the accuracy of approximation increases as $n \to \infty$. This is in agreement with the observation made in the beginning of section 2.8: the cusp anomalous dimension in the toy model $\Gamma_{\rm cusp}^{\rm (toy)}(g)$ is close numerically to the exact expression $\Gamma_{\rm cusp}(g)$. In addition, the same property suggests that the coefficients in the strong coupling expansion of $\Gamma_{\rm cusp}^{\rm (toy)}(g)$ and $\Gamma_{\rm cusp}(g)$ should have the same large order behavior. It was found in [25] that the expansion coefficients on the right-hand side of (2.63) grow at higher orders as $\Gamma_{\rm cusp}(g) \sim \sum_k \Gamma\left(k+\frac12\right)\alpha^k$. It is straightforward to verify using (2.55) and (2.60) that the expansion coefficients of $C_0(\alpha)$ in the toy model have the same behavior. This suggests that nonperturbative corrections to $\Gamma_{\rm cusp}(g)$ and $\Gamma_{\rm cusp}^{\rm (toy)}(g)$

are parameterized by the same scale Λ^2 defined in (2.53). Indeed, we will show this in the following section by explicit calculation.

We demonstrated in this section that nonperturbative corrections in the toy model follow unambiguously from the exact solution (2.50). In the following section, we will extend analysis to the cusp anomalous dimension and work out the strong coupling expansion of $\Gamma_{\rm cusp}(g)/(2g)$ analogous to (2.59).

3. Solving the quantization conditions

Let us now solve the quantization conditions (2.45) for the cusp anomalous dimension. Relation (2.45) involves two sets of functions. The functions $V_0(t)$ and $V_1(t)$ are given by the Whittaker function of the first kind (see equation (D.2)). At the same time, the functions $f_0(t)$ and $f_1(t)$ are defined in (2.40) and they depend on the (infinite) set of expansion coefficients $c(g)$ and $c_\pm(n, g)$. Having determined these coefficients from the quantization conditions (2.45), we can then compute the cusp anomalous dimension for arbitrary coupling with the help of (2.47).

We expect that at strong coupling, the resulting expression for $\Gamma_{\rm cusp}(g)$ will have form (1.2). Examining (2.47), we observe that the dependence on the coupling resides both in the expansion coefficients and in the functions $U_0^\pm(4\pi g)$. The latter are given by the Whittaker functions of the second kind (see equation (D.7)) and, as such, they are given by Borel-summable sign-alternating asymptotic series in $1/g$. Therefore, nonperturbative corrections to the cusp anomalous dimension (2.47) could only come from the coefficients $c_\pm(n, g)$ and $c(g)$.

3.1. Quantization conditions

Let us replace $f_0(t)$ and $f_1(t)$ in (2.45) by their explicit expressions (2.40) and rewrite the quantization conditions (2.45) as

$$V_0(4\pi g x_\ell) + c(g) V_1(4\pi g x_\ell) = \sum_{n\geqslant 1} [c_+(n, g) A_+(n, x_\ell) + c_-(n, g) A_-(n, x_\ell)] , \tag{3.1}$$

where $x_\ell = \ell - \frac{1}{4}$ (with $\ell = 0, \pm1, \pm2, \ldots$) and the notation was introduced for

$$A_\pm(n, x_\ell) = \frac{n V_1(4\pi g x_\ell) U_0^\pm(4\pi n g) + x_\ell V_0(4\pi g x_\ell) U_1^\pm(4\pi n g)}{n \mp x_\ell}. \tag{3.2}$$

Relation (3.1) provides an infinite system of linear equations for $c_\pm(g, n)$ and $c(g)$. The coefficients in this system depend on $V_{0,1}(4\pi g x_\ell)$ and $U_{0,1}^\pm(4\pi n g)$ which are known functions defined in appendix D. We would like to stress that relation (3.1) holds for arbitrary $g > 0$.

Let us show that the quantization conditions (3.1) lead to $c(g) = 0$ for arbitrary coupling. To this end, we examine (3.1) for $|x_\ell| \gg 1$. In this limit, for $g =$ fixed we are allowed to replace the functions $V_0(4\pi g x_\ell)$ and $V_1(4\pi g x_\ell)$ on both sides of (3.1) by their asymptotic behavior at infinity. Making use of (D.10) and (D.12), we find for $|x_\ell| \gg 1$

$$r(x_\ell) \equiv \frac{V_1(4\pi g x_\ell)}{V_0(4\pi g x_\ell)} = \begin{cases} -16\pi g x_\ell + \cdots, & (x_\ell < 0) \\ \frac{1}{2} + \cdots, & (x_\ell > 0), \end{cases} \tag{3.3}$$

where the ellipses denote terms suppressed by powers of $1/(g x_\ell)$ and $e^{-8\pi g |x_\ell|}$. We divide both sides of (3.1) by $V_1(4\pi g x_\ell)$ and observe that for $x_\ell \to -\infty$, the first term on the left-hand side of (3.1) is subleading and can be safely neglected. In a similar manner, one has

Table 1. Comparison of the numerical value of $\Gamma_{\rm cusp}(g)/(2g)$ found from (3.1) and (2.47) for $n_{\max} = 40$ with the exact one [25, 45] for different values of the coupling constant g.

g	0.1	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8
Numer.	0.1976	0.3616	0.5843	0.7096	0.7825	0.8276	0.8576	0.8787	0.8944	0.9065
Exact	0.1939	0.3584	0.5821	0.7080	0.7813	0.8267	0.8568	0.8781	0.8938	0.9059

Table 2. Dependence of $\Gamma_{\rm cusp}(g)/(2g)$ on the truncation parameter $n_{\max}$ for $g = 1$ and $c(g) = 0$. The last column describes the exact result.

$n_{\max}$	10	20	30	40	50	60	70	∞
Numer.	0.8305	0.8286	0.8279	0.8276	0.8274	0.8273	0.8272	0.8267

$A_\pm(n, x_\ell)/V_1(4\pi g x_\ell) = O(1/x_\ell)$ for fixed n on the right-hand side of (3.1). Therefore, going to the limit $x_\ell \to -\infty$ on both sides of (3.1) we get

$$c(g) = 0 \tag{3.4}$$

for arbitrary g. We verify in appendix A by explicit calculation that this relation indeed holds at weak coupling.

Arriving at (3.4), we tacitly assumed that the sum over n in (3.1) remains finite in the limit $x_\ell \to -\infty$. Taking into account large n behavior of the functions $U_0^\pm(4\pi n g)$ and $U_1^\pm(4\pi n g)$ (see equation (D.12)), we obtain that this condition translates into the following condition for asymptotic behavior of the coefficients at large n:

$$c_+(n, g) = o(n^{1/4}), \qquad c_-(n, g) = o(n^{-1/4}). \tag{3.5}$$

These relations also ensure that the sum in expression (2.47) for the cusp anomalous dimension is convergent.

3.2. Numerical solution

To begin with, let us solve the infinite system of linear equations (3.1) numerically. In order to verify (3.4), we decided to do it in two steps: we first solve (3.1) for $c_\pm(n, g)$ assuming $c(g) = 0$ and, then, repeat the same analysis by relaxing condition (3.4) and treating $c(g)$ as unknown.

For $c(g) = 0$, we truncate the infinite sums on the right-hand side of (3.1) at some large $n_{\max}$ and, then, use (3.1) for $\ell = 1 - n_{\max}, \dots, n_{\max}$ to find numerical values of $c_\pm(n, g)$ with $1 \leqslant n \leqslant n_{\max}$ for given coupling g. Substituting the resulting expressions for $c_\pm(n, g)$ into (2.47), we compute the cusp anomalous dimension. Taking the limit $n_{\max} \to \infty$, we expect to recover the exact result. Results of our analysis are summarized in two tables. Table 1 shows the dependence of the cusp anomalous dimension on the coupling constant. Table 2 shows the dependence of the cusp anomalous dimension on the truncation parameter $n_{\max}$ for fixed coupling.

For $c(g)$ arbitrary, we use (3.1) for $\ell = -n_{\max}, \dots, n_{\max}$ to find numerical values of $c(g)$ and $c_\pm(n, g)$ with $1 \leqslant n \leqslant n_{\max}$ for given coupling g. In this manner, we compute $\Gamma_{\rm cusp}(g)/(2g)$ and $c(g)$ and, then, compare them with the exact expressions corresponding to $n_{\max} \to \infty$. For the cusp anomalous dimension, our results for $\Gamma_{\rm cusp}(g)/(2g)$ are in remarkable agreement with the exact expression. Namely, for $n_{\max} = 40$ their difference equals 5.480×10^{-6} for $g = 1$ and it decreases down to 8.028×10^{-7} for $g = 1.8$. The reason

Table 3. Dependence of $c(g)$ on the truncation parameter $n_{\max}$ for $g = 1$ derived from the quantization condition (3.1).

$n_{\max}$	10	20	30	40	50	60	70	∞
$-c(g)$	0.0421	0.0357	0.0323	0.0301	0.0285	0.0272	0.0262	0

why the agreement is better compared to the $c(g) = 0$ case (see table 1) is that $c(g)$ takes effectively into account a reminder of the sum on the right-hand side of (3.1) corresponding to $n > n_{\max}$. The dependence of the expression obtained for $c(g)$ on the truncation parameter $n_{\max}$ is shown in table 3. We observe that, in agreement with (3.4), $c(g)$ vanishes as $n_{\max} \to \infty$.

Our numerical analysis shows that the cusp anomalous dimension (2.47) can be determined from the quantization conditions (3.1) and (3.4) for arbitrary coupling g. In distinction with the toy model (2.50), the resulting expression for $\Gamma_{\text{cusp}}(g)$ does not admit a closed form representation. Still, as we will show in the following subsection, the quantization conditions (3.1) can be solved analytically for $g \gg 1$ leading to asymptotic expansion for the cusp anomalous dimension at strong coupling.

3.3. Strong coupling solution

Let us divide both sides of (3.1) by $V_0(4\pi g x_\ell)$ and use (3.4) to get (for $x_\ell = \ell - \frac{1}{4}$ and $\ell \in \mathbb{Z}$)

$$1 = \sum_{n\geqslant 1} c_+(n,g)\left[\frac{nU_0^+(4\pi ng)r(x_\ell) + U_1^+(4\pi ng)x_\ell}{n - x_\ell}\right] + \sum_{n\geqslant 1} c_-(n,g)\left[\frac{nU_0^-(4\pi ng)r(x_\ell) + U_1^-(4\pi ng)x_\ell}{n + x_\ell}\right], \tag{3.6}$$

where the function $r(x_\ell)$ was defined in (3.3).

Let us now examine the large g asymptotics of the coefficient functions accompanying $c_\pm(n,g)$ on the right-hand side of (3.6). The functions $U_0^\pm(4\pi ng)$ and $U_1^\pm(4\pi ng)$ admit asymptotic expansion in $1/g$ given by (D.12). For the function $r(x_\ell)$, the situation is different. As follows from its definition, equations (3.3) and (D.10), large g expansion of $r(x_\ell)$ runs in two parameters: perturbative $1/g$ and nonperturbative exponentially small parameter $\Lambda^2 \sim g^{1/2}\,\mathrm{e}^{-2\pi g}$ which we already encountered in the toy model (equation (2.53)). Moreover, we deduce from (3.3) and (D.10) that the leading nonperturbative correction to $r(x_\ell)$ scales as

$$\delta r(x_\ell) = O(\Lambda^{|8\ell-2|}), \qquad \left(x_\ell = \ell - \tfrac{1}{4}, \ell \in \mathbb{Z}\right), \tag{3.7}$$

so that the power of Λ grows with ℓ. We observe that $O(\Lambda^2)$ corrections are only present in $r(x_\ell)$ for $\ell = 0$. Therefore, as far as the leading $O(\Lambda^2)$ correction to the solutions to (3.6) are concerned, we are allowed to neglect nonperturbative (Λ^2-dependent) corrections to $r(x_\ell)$ on the right-hand side of (3.6) for $\ell \neq 0$ and retain them for $\ell = 0$ only.

Since the coefficient functions in the linear equations (3.6) admit a double series expansion in powers of $1/g$ and Λ^2, we expect that the same should be true for their solutions $c_\pm(n,g)$. Let us determine the first few terms of this expansion using the following ansatz:

$$c_\pm(n,g) = (8\pi gn)^{\pm 1/4}\left\{\left[a_\pm(n) + \frac{b_\pm(n)}{4\pi g} + \cdots\right] + \Lambda^2\left[\alpha_\pm(n) + \frac{\beta_\pm(n)}{4\pi g} + \cdots\right] + O(\Lambda^4)\right\}, \tag{3.8}$$

where Λ^2 is a nonperturbative parameter defined in (2.53):

$$\Lambda^2 = \sigma(2\pi g)^{1/2}\,\mathrm{e}^{-2\pi g}\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)}, \tag{3.9}$$

and the ellipses denote terms suppressed by powers of $1/g$. Here, the functions $a_\pm(n), b_\pm(n), \ldots$ are assumed to be g-independent. We recall that the functions $c_\pm(n,g)$ have to verify relation (3.5). This implies that the functions $a_\pm(n), b_\pm(n), \ldots$ should vanish as $n\to\infty$. To determine them, we substitute (3.8) into (3.6) and compare the coefficients in front of powers of $1/g$ and Λ^2 on both sides of (3.6).

3.3.1. Perturbative corrections. Let us start with the 'perturbative', Λ^2-independent part of (3.8) and compute the functions $a_\pm(n)$ and $b_\pm(n)$.

To determine $a_\pm(n)$, we substitute (3.8) into (3.6), replace the functions $U^\pm_{0,1}(4\pi gn)$ and $r(x_\ell)$ by their large g asymptotic expansion, equations (D.12) and (3.3), respectively, neglect corrections in Λ^2 and compare the leading $O(g^0)$ terms on both sides of (3.6). In this way, we obtain from (3.6) the following relations for $a_\pm(n)$ (with $x_\ell = \ell - \frac{1}{4}$):

$$\begin{aligned} &2x_\ell\Gamma\left(\frac{5}{4}\right)\sum_{n\geqslant 1}\frac{a_+(n)}{n-x_\ell} = 1, \qquad (\ell\geqslant 1)\\ &-2x_\ell\Gamma\left(\frac{3}{4}\right)\sum_{n\geqslant 1}\frac{a_-(n)}{n+x_\ell} = 1, \qquad (\ell\leqslant 0).\end{aligned} \tag{3.10}$$

One can verify that the solutions to this system satisfying $a_\pm(n)\to 0$ for $n\to\infty$ have the form

$$\begin{aligned} a_+(n) &= \frac{2\Gamma\left(n+\frac{1}{4}\right)}{\Gamma(n+1)\Gamma^2\left(\frac{1}{4}\right)},\\ a_-(n) &= \frac{\Gamma\left(n+\frac{3}{4}\right)}{2\Gamma(n+1)\Gamma^2\left(\frac{3}{4}\right)}.\end{aligned} \tag{3.11}$$

In a similar manner, we compare the subleading $O(1/g)$ terms on both sides of (3.6) and find that the functions $b_\pm(n)$ satisfy the following relations (with $x_\ell = \ell - \frac{1}{4}$):

$$\begin{aligned} &2x_\ell\Gamma\left(\frac{5}{4}\right)\sum_{n\geqslant 1}\frac{b_+(n)}{n-x_\ell} = -\frac{3}{32x_\ell} - \frac{3\pi}{64} - \frac{15}{32}\ln 2, \qquad (\ell\geqslant 1)\\ &-2x_\ell\Gamma\left(\frac{3}{4}\right)\sum_{n\geqslant 1}\frac{b_-(n)}{n+x_\ell} = -\frac{5}{32x_\ell} - \frac{5\pi}{64} + \frac{9}{32}\ln 2, \qquad (\ell\leqslant 0),\end{aligned} \tag{3.12}$$

where on the right-hand side we made use of (3.11). The solutions to these relations are

$$b_+(n) = -a_+(n)\left(\frac{3\ln 2}{4} + \frac{3}{32n}\right), \qquad b_-(n) = a_-(n)\left(\frac{3\ln 2}{4} + \frac{5}{32n}\right). \tag{3.13}$$

It is straightforward to extend the analysis to subleading perturbative corrections to $c_\pm(n,g)$.

Let us substitute (3.8) into expression (2.47) for the cusp anomalous dimension. Taking into account the identities (D.12), we find the 'perturbative' contribution to $\Gamma_{\mathrm{cusp}}(g)$ as

$$\begin{aligned}\Gamma_{\mathrm{cusp}}(g) = 2g - \sum_{n\geqslant 1}(2\pi n)^{-1}\Bigg[&\Gamma\left(\frac{5}{4}\right)\left(a_+(n)+\frac{b_+(n)}{4\pi g}+\cdots\right)\left(1-\frac{5}{128\pi gn}+\ldots\right)\\ &+\Gamma\left(\frac{3}{4}\right)\left(a_-(n)+\frac{b_-(n)}{4\pi g}+\cdots\right)\left(1+\frac{3}{128\pi gn}+\cdots\right)\Bigg] + O(\Lambda^2).\end{aligned} \tag{3.14}$$

Replacing $a_\pm(n)$ and $b_\pm(n)$ by their expressions (3.11) and (3.13), we find after some algebra

$$\Gamma_{\text{cusp}}(g) = 2g\left[1 - \frac{3\ln 2}{4\pi g} - \frac{\mathrm{K}}{16\pi^2 g^2} + O(1/g^3)\right] + O(\Lambda^2), \tag{3.15}$$

where K is the Catalan number. This relation is in agreement with the known result obtained both in $\mathcal{N} = 4$ SYM theory [25, 26] and in string theory [29].

3.3.2. Nonperturbative corrections. Let us now compute the leading $O(\Lambda^2)$ nonperturbative correction to the coefficients $c_\pm(n, g)$. According to (3.8), it is described by the functions $\alpha_\pm(n)$ and $\beta_\pm(n)$. To determine them from (3.6), we have to retain in $r(x_\ell)$ the corrections proportional to Λ^2. As was already explained, they only appear for $\ell = 0$. Combining together relations (3.3), (D.10) and (D.12), we find after some algebra

$$\delta r(x_\ell) = -\delta_{\ell,0}\Lambda^2\left[4\pi g - \tfrac{5}{4} + O(g^{-1})\right] + O(\Lambda^4). \tag{3.16}$$

Let us substitute this relation into (3.6) and equate to zero the coefficient in front of Λ^2 on the right-hand side of (3.6). This coefficient is given by series in $1/g$ and, examining the first two terms, we obtain the relations for the functions $\alpha_\pm(n)$ and $\beta_\pm(n)$.

In this way, we find that the leading functions $\alpha_\pm(n)$ satisfy the relations (with $x_\ell = \ell - \frac{1}{4}$)

$$\begin{aligned} &2x_\ell\Gamma\left(\frac{5}{4}\right)\sum_{n\geqslant 1}\frac{\alpha_+(n)}{n - x_\ell} = 0, && (\ell \geqslant 1) \\ &-2x_\ell\Gamma\left(\frac{3}{4}\right)\sum_{n\geqslant 1}\frac{\alpha_-(n)}{n + x_\ell} = \frac{\pi}{2\sqrt{2}}\delta_{\ell,0}, && (\ell \leqslant 0), \end{aligned} \tag{3.17}$$

where on the right-hand side we applied (3.11). The solution to (3.17) satisfying $\alpha_\pm(n) \to 0$ as $n \to \infty$ reads as

$$\begin{aligned} &\alpha_+(n) = 0, \\ &\alpha_-(n) = a_-(n-1), \end{aligned} \tag{3.18}$$

with $a_-(n)$ defined in (3.11). For subleading functions $\beta_\pm(n)$, we have similar relations:

$$\begin{aligned} &2x_\ell\Gamma\left(\frac{5}{4}\right)\sum_{n\geqslant 1}\frac{\beta_+(n)}{n - x_\ell} = -\frac{1}{2}, && (\ell \geqslant 1) \\ &-2x_\ell\Gamma\left(\frac{3}{4}\right)\sum_{n\geqslant 1}\frac{\beta_-(n)}{n + x_\ell} = -\frac{1}{8} + \frac{3\pi}{16\sqrt{2}}(1 - 2\ln 2)\delta_{\ell,0}, && (\ell \leqslant 0). \end{aligned} \tag{3.19}$$

In a close analogy with (3.13), the solutions to these relations can be written in terms of leading-order functions $a_\pm(n)$ defined in (3.11):

$$\begin{aligned} &\beta_+(n) = -\frac{1}{2}a_+(n), \\ &\beta_-(n) = a_-(n-1)\left(\frac{1}{4} - \frac{3\ln 2}{4} + \frac{1}{32n}\right). \end{aligned} \tag{3.20}$$

It is straightforward to extend the analysis and compute subleading $O(\Lambda^2)$ corrections to (3.8).

Relation (3.8) supplemented with (3.11), (3.13), (3.18) and (3.20) defines the solution to the quantization condition (3.6) to leading order in both perturbative, $1/g$, and nonperturbative,

Λ^2, expansion parameters. We are now ready to compute nonperturbative correction to the cusp anomalous dimension (2.47). Substituting (3.8) into (2.47), we obtain

$$\delta\Gamma_{\text{cusp}}(g) = -\Lambda^2 \sum_{n\geqslant 1}(2\pi n)^{-1}\left[\Gamma\left(\frac{5}{4}\right)\left(\alpha_+(n) + \frac{\beta_+(n)}{4\pi g} + \cdots\right)\left(1 - \frac{5}{128\pi g n} + \cdots\right)\right.$$
$$\left. + \Gamma\left(\frac{3}{4}\right)\left(\alpha_-(n) + \frac{\beta_-(n)}{4\pi g} + \cdots\right)\left(1 + \frac{3}{128\pi g n} + \cdots\right)\right] + O(\Lambda^4). \tag{3.21}$$

We replace $\alpha_\pm(n)$ and $\beta_\pm(n)$ by their explicit expressions (3.18) and (3.20), respectively, evaluate the sums and find

$$\delta\Gamma_{\text{cusp}}(g) = -\frac{\Lambda^2}{\pi}\left[1 + \frac{3 - 6\ln 2}{16\pi g} + O(1/g^2)\right] + O(\Lambda^4), \tag{3.22}$$

with Λ^2 defined in (3.9).

Relations (3.15) and (3.22) describe, correspondingly, perturbative and nonperturbative corrections to the cusp anomalous dimension. Let us define a new nonperturbative parameter m_{cusp}^2 whose meaning will be clear in a moment:

$$m_{\text{cusp}}^2 = \frac{4\sqrt{2}}{\pi\sigma}\Lambda^2\left[1 + \frac{3 - 6\ln 2}{16\pi g} + O(1/g^2)\right] + O(\Lambda^4). \tag{3.23}$$

Then, expressions (3.15) and (3.22) obtained for the cusp anomalous dimension take the form

$$\Gamma_{\text{cusp}}(g) = \left[2g - \frac{3\ln 2}{2\pi} - \frac{\mathrm{K}}{8\pi^2 g} + O(1/g^2)\right] - \frac{\sigma}{4\sqrt{2}} m_{\text{cusp}}^2 + O\left(m_{\text{cusp}}^4\right). \tag{3.24}$$

We recall that another nonperturbative parameter was already introduced in section 2.1 as defining the mass gap $m_{O(6)}$ in the O(6) model. We will show in the following section that the two scales, m_{cusp} and $m_{O(6)}$, coincide to any order in $1/g$.

4. Mass scale

The cusp anomalous dimension controls the leading logarithmic scaling behavior of the anomalous dimensions (1.1) in the double scaling limit $L, N \to \infty$ and $j = L/\ln N =$ fixed. The subleading corrections to this behavior are described by the scaling function $\epsilon(j, g)$. At strong coupling, this function coincides with the energy density of the ground state of the bosonic O(6) model (1.3). The mass gap in this model $m_{O(6)}$ is given by expression (2.9) which involves the functions $\Gamma_\pm(t)$ constructed in section 2.

4.1. General expression

Let us apply (2.9) and compute the mass gap $m_{O(6)}$ at strong coupling. At large g, the integral in (2.9) receives a dominant contribution from $t \sim g$. In order to evaluate (2.9), it is convenient to change the integration variable as $t \to 4\pi g i t$:

$$m_{O(6)} = \frac{8\sqrt{2}}{\pi^2}\,\mathrm{e}^{-\pi g} - \frac{8g}{\pi}\,\mathrm{e}^{-\pi g}\mathrm{Re}\left[\int_0^{-\mathrm{i}\infty} \mathrm{d}t\,\mathrm{e}^{-4\pi g t - \mathrm{i}\pi/4}\frac{\Gamma(4\pi g i t)}{t + \frac{1}{4}}\right], \tag{4.1}$$

where the integration goes along the imaginary axis. We find from (2.39) that $\Gamma(4\pi g i t)$ takes the form

$$\Gamma(4\pi g i t) = f_0(4\pi g t)V_0(4\pi g t) + f_1(4\pi g t)V_1(4\pi g t), \tag{4.2}$$

where $V_{0,1}(4\pi gt)$ are given by the Whittaker functions of the first kind, equation (D.2), and $f_{0,1}(4\pi gt)$ admit the following representation (see equations (2.40) and (3.4)):

$$
\begin{aligned}
f_0(4\pi gt) &= \sum_{n\geqslant 1} t\left[c_+(n,g)\frac{U_1^+(4\pi ng)}{n-t} + c_-(n,g)\frac{U_1^-(4\pi ng)}{n+t}\right] - 1,\\
f_1(4\pi gt) &= \sum_{n\geqslant 1} n\left[c_+(n,g)\frac{U_0^+(4\pi ng)}{n-t} + c_-(n,g)\frac{U_0^-(4\pi ng)}{n+t}\right].
\end{aligned}
\tag{4.3}
$$

Here, the functions $U_{0,1}^{\pm}(4\pi ng)$ are expressed in terms of the Whittaker functions of the first kind, equation (D.7), and the expansion coefficients $c_\pm(n,g)$ are solutions to the quantization conditions (2.45).

Replacing $\Gamma(4\pi git)$ in (4.1) by its expression (4.2), we evaluate the t-integral and find after some algebra (see appendix E for details) [38]

$$
m_{O(6)} = -\frac{16\sqrt{2}}{\pi} g \mathrm{e}^{-\pi g}[f_0(-\pi g)U_0^-(\pi g) + f_1(-\pi g)U_1^-(\pi g)]. \tag{4.4}
$$

This relation can be further simplified with the help of the quantization conditions (2.45). For $\ell = 0$, we obtain from (2.45) that $f_0(-\pi g)V_0(-\pi g) + f_1(-\pi g)V_1(-\pi g) = 0$. Together with the Wronskian relation for the Whittaker functions (D.8), this leads to the following remarkable relation for the mass gap:

$$
m_{O(6)} = \frac{16\sqrt{2}}{\pi^2}\frac{f_1(-\pi g)}{V_0(-\pi g)}. \tag{4.5}
$$

It is instructive to compare this relation with a similar relation (2.46) for the cusp anomalous dimension. We observe that both quantities involve the same function $f_1(4\pi gt)$ but evaluated for different values of its argument, that is, $t = -1/4$ for the mass gap and $t = 0$ for the cusp anomalous dimension. As a consequence, there are no reasons to expect that the two functions, $m(g)$ and $\Gamma_{\mathrm{cusp}}(g)$, could be related to each other in a simple way. Nevertheless, we will demonstrate in this subsection that $m_{O(6)}^2$ determines the leading nonperturbative correction to $\Gamma_{\mathrm{cusp}}(g)$ at strong coupling.

4.2. Strong coupling expansion

Let us now determine the strong coupling expansion of functions (4.3). We replace coefficients $c_\pm(n,g)$ in (4.3) by their expression (3.8) and take into account the results obtained for the functions $a_\pm, b_\pm, \ldots$ (equations (3.11), (3.13), (3.18) and (3.20)). In addition, we replace in (4.3) the functions $U_{0,1}^{\pm}(4\pi ng)$ by their strong coupling expansion (D.12). We recall that the coefficients $c_\pm(n,g)$ admit the double series expansion (3.8) in powers of $1/g$ and $\Lambda^2 \sim \mathrm{e}^{-2\pi g}$ (equation (3.9)). As a consequence, the functions $f_0(4\pi gt)$ and $f_1(4\pi gt)$ have the form

$$
f_n(4\pi gt) = f_n^{(\mathrm{PT})}(4\pi gt) + \delta f_n(4\pi gt), \qquad (n = 0, 1), \tag{4.6}
$$

where $f_n^{(\mathrm{PT})}$ is given by asymptotic (non-Borel summable) series in $1/g$ and δf_n takes into account the nonperturbative corrections in Λ^2.

Evaluating sums on the right-hand side of (4.3), we find that $f_0(4\pi gt)$ and $f_1(4\pi gt)$ can be expressed in terms of two sums involving functions $a_\pm(n)$ defined in (3.11):

$$
\begin{aligned}
2\Gamma\left(\frac{5}{4}\right)\sum_{n\geqslant 1}\frac{a_+(n)}{t-n} &= \frac{1}{t}\left[\frac{\Gamma\left(\frac{3}{4}\right)\Gamma(1-t)}{\Gamma\left(\frac{3}{4}-t\right)} - 1\right],\\
2\Gamma\left(\frac{3}{4}\right)\sum_{n\geqslant 1}\frac{a_-(n)}{t+n} &= \frac{1}{t}\left[\frac{\Gamma\left(\frac{1}{4}\right)\Gamma(1+t)}{\Gamma\left(\frac{1}{4}+t\right)} - 1\right].
\end{aligned}
\tag{4.7}
$$

Going through calculation of (4.3), we find after some algebra that perturbative corrections to $f_0(4\pi gt)$ and $f_1(4\pi gt)$ are given by linear combinations of the ratios of Euler gamma functions:

$$\begin{aligned}
f_0^{\text{(PT)}}(4\pi gt) &= -\frac{\Gamma\left(\frac{3}{4}\right)\Gamma(1-t)}{\Gamma\left(\frac{3}{4}-t\right)} \\
&\quad + \frac{1}{4\pi g}\left[\left(\frac{3\ln 2}{4}+\frac{1}{8t}\right)\frac{\Gamma\left(\frac{3}{4}\right)\Gamma(1-t)}{\Gamma\left(\frac{3}{4}-t\right)} - \frac{\Gamma\left(\frac{1}{4}\right)\Gamma(1+t)}{8t\Gamma\left(\frac{1}{4}+t\right)}\right] + O(g^{-2}),\\
f_1^{\text{(PT)}}(4\pi gt) &= \frac{1}{4\pi g}\left[\frac{\Gamma\left(\frac{1}{4}\right)\Gamma(1+t)}{4t\Gamma\left(\frac{1}{4}+t\right)} - \frac{\Gamma\left(\frac{3}{4}\right)\Gamma(1-t)}{4t\Gamma\left(\frac{3}{4}-t\right)}\right]\\
&\quad - \frac{1}{(4\pi g)^2}\left[\frac{\Gamma\left(\frac{1}{4}\right)\Gamma(1+t)}{4t\Gamma\left(\frac{1}{4}+t\right)}\left(\frac{1}{4t}-\frac{3\ln 2}{4}\right)\right.\\
&\quad \left. - \frac{\Gamma\left(\frac{3}{4}\right)\Gamma(1-t)}{4t\Gamma\left(\frac{3}{4}-t\right)}\left(\frac{1}{4t}+\frac{3\ln 2}{4}\right)\right] + O(g^{-3}).
\end{aligned} \tag{4.8}$$

Note that $f_1(t)$ is suppressed by factor $1/(4\pi g)$ compared to $f_0(t)$. In a similar manner, we compute nonperturbative corrections to (4.6):

$$\begin{aligned}
\delta f_0(4\pi gt) &= \Lambda^2\left\{\frac{1}{4\pi g}\left[\frac{\Gamma\left(\frac{3}{4}\right)\Gamma(1-t)}{2\Gamma\left(\frac{3}{4}-t\right)} - \frac{\Gamma\left(\frac{5}{4}\right)\Gamma(1+t)}{2\Gamma\left(\frac{5}{4}+t\right)}\right] + O(g^{-2})\right\} + \cdots,\\
\delta f_1(4\pi gt) &= \Lambda^2\left\{\frac{1}{4\pi g}\frac{\Gamma\left(\frac{5}{4}\right)\Gamma(1+t)}{\Gamma\left(\frac{5}{4}+t\right)} + \frac{1}{(4\pi g)^2}\left[\frac{\Gamma\left(\frac{3}{4}\right)\Gamma(1-t)}{8t\Gamma\left(\frac{3}{4}-t\right)}\right.\right.\\
&\quad \left.\left. - \frac{\Gamma\left(\frac{5}{4}\right)\Gamma(1+t)}{\Gamma\left(\frac{5}{4}+t\right)}\left(\frac{1}{8t}+\frac{3}{4}\ln 2-\frac{1}{4}\right)\right] + O(g^{-3})\right\} + \cdots,
\end{aligned} \tag{4.9}$$

where the ellipses denote $O(\Lambda^4)$ terms.

Substituting (4.8) and (4.9) into (4.2), we obtain the strong coupling expansion of the function $\Gamma(4\pi igt)$. To verify the expressions obtained, we apply (2.46) to calculate the cusp anomalous dimension:

$$\Gamma_{\text{cusp}}(g) = 2g - 4gf_1^{\text{(PT)}}(0) - 4g\delta f_1(0). \tag{4.10}$$

Replacing $f_1^{\text{(PT)}}(0)$ and $\delta f_1(0)$ by their expressions, equations (4.8) and (4.9), we obtain

$$\Gamma_{\text{cusp}}(g) = 2g\left[1-\frac{3\ln 2}{4\pi g}-\frac{\mathrm{K}}{(4\pi g)^2}+\cdots\right] - \frac{\Lambda^2}{\pi}\left[1+\frac{3-6\ln 2}{16\pi g}+\cdots\right] + O(\Lambda^4), \tag{4.11}$$

in a perfect agreement with (3.15) and (3.22), respectively.

Let us obtain the strong coupling expansion of the mass gap (4.5). We replace $V_0(-\pi g)$ by its asymptotic series, equations (D.14) and (D.12), and take into account (4.8) and (4.9) to get

$$\begin{aligned}
m_{O(6)} &= \frac{\sqrt{2}}{\Gamma\left(\frac{5}{4}\right)}(2\pi g)^{1/4}\,\mathrm{e}^{-\pi g}\left\{\left[1+\frac{3-6\ln 2}{32\pi g}+\frac{-63+108\ln 2-108(\ln 2)^2+16\mathrm{K}}{2048(\pi g)^2}+\cdots\right]\right.\\
&\quad \left. - \frac{\Lambda^2}{8\pi g}\left[1-\frac{15-6\ln 2}{32\pi g}+\cdots\right] + O(\Lambda^4)\right\}.
\end{aligned} \tag{4.12}$$

Here, in order to determine $O(1/g^2)$ and $O(\Lambda^2/g^2)$ terms inside the curly brackets, we computed in addition the subleading $O(g^{-3})$ corrections to $f_1^{\text{(PT)}}$ and δf_1 in equations (4.8) and (4.9), respectively. The leading $O(1/g)$ correction to $m_{O(6)}$ (the second term inside the first square bracket on the right-hand side of (4.12)) is in agreement with both analytical [35, 38] and numerical calculations [36].

We are now ready to clarify the origin of the 'substitution rule' (2.66) that establishes the relation between the cusp anomalous dimension in the toy model and the exact solution. To this end, we compare the expressions for the functions $f_n(4\pi gt)$ given by (4.6), (4.8) and (4.9) with those in the toy model (equations (2.42) and (2.48))[9]. It is straightforward to verify that upon the substitution (2.66) and (2.64), the two sets of functions coincide up to an overall t-dependent factor[10]:

$$f_n(4\pi gt)\frac{\Gamma\left(\frac{3}{4}-t\right)}{\Gamma\left(\frac{3}{4}\right)\Gamma(1-t)} \quad\rightarrow\quad f_n^{(\mathrm{toy})}(4\pi gt), \qquad (n=0,1). \tag{4.13}$$

Since the cusp anomalous dimension (2.46) is determined by the f_1-function evaluated at $t=0$, the additional factor does not affect its value.

4.3. Nonperturbative corrections to the cusp anomalous dimension

Relation (4.12) defines strong coupling corrections to the mass gap. In a close analogy with the cusp anomalous dimension (4.11), it runs in two parameters: perturbative $1/g$ and nonperturbative Λ^2. We would like to stress that the separation of the corrections to $m_{O(6)}$ into perturbative and nonperturbative ones is ambiguous since the 'perturbative' series inside the square brackets on the right-hand side of (4.12) is non-Borel summable and, therefore, it suffers from Borel ambiguity. It is only the sum of perturbative and nonperturbative corrections that is an unambiguously defined function of the coupling constant. In distinction with the mass scale $m_{O(6)}$, definition (2.53) of the nonperturbative scale Λ^2 involves a complex parameter σ whose value depends on the prescription employed to regularize the singularities of the 'perturbative' series.

To illustrate the underlying mechanism of the cancellation of Borel ambiguity inside $m_{O(6)}$, let us examine the expression for the mass gap (4.5) in the toy model. As was already explained in section 2.8, the toy model captures the main features of the exact solution at strong coupling and, at the same time, it allows us to obtain expressions for various quantities in a closed analytical form. The mass gap in the toy model is given by relation (4.5) with $f_1(-\pi g)$ replaced with $f_1^{(\mathrm{toy})}(-\pi g)$ defined in (2.42) and (2.48). In this way, we obtain

$$m_{\mathrm{toy}} = \frac{16\sqrt{2}}{\pi^2}\frac{f_1^{(\mathrm{toy})}(-\pi g)}{V_0(-\pi g)} = \frac{16\sqrt{2}}{\pi^2}\frac{1}{V_1(-\pi g)}. \tag{4.14}$$

Here, $V_1(-\pi g)$ is an entire function of the coupling constant (see equation (2.41)). Its large g asymptotic expansion can be easily deduced from (D.16) and it involves the nonperturbative parameter Λ^2.

Making use of (2.52), we obtain from (4.14)

$$\begin{aligned} m_{\mathrm{toy}} = \frac{4}{\pi\Gamma\left(\frac{5}{4}\right)}(2\pi g)^{1/4}\,\mathrm{e}^{-\pi g}\bigg\{&\left[1-\frac{1}{32\pi g}-\frac{23}{2048(\pi g)^2}+\cdots\right] \\ &-\frac{\Lambda^2}{8\pi g}\left[1-\frac{11}{32\pi g}+\cdots\right]+O(\Lambda^4)\bigg\}, \end{aligned} \tag{4.15}$$

[9] It worth mentioning that the functions $f_0^{(\mathrm{toy})}$ and $f_1^{(\mathrm{toy})}$ in the toy model are, in fact, t-independent.

[10] Roughly speaking, this substitution simplifies the complicated structure of poles and zeros of the exact solution, equations (4.8) and (4.9), encoded in the ratio of the gamma functions to match simple analytical properties of the same functions in the toy model (compare (2.13) and (2.16)).

where the ellipses denote terms with higher power of $1/g$. By construction, $m_{\rm toy}$ is an unambiguous function of the coupling constant whereas the asymptotic series inside the square brackets are non-Borel summable. It is easy to verify that 'perturbative' corrections to $m_{\rm toy}^2$ are described by the asymptotic series $C_2(\alpha)$ given by (2.61). Together with (2.59), this allows us to identify the leading nonperturbative correction to (2.59) in the toy model as

$$\delta\Gamma_{\rm cusp}^{\rm (toy)} = -\frac{\Lambda^2}{\pi} C_2(\alpha) + O(\Lambda^4) = -\frac{\pi^2}{32\sqrt{2}}\sigma m_{\rm toy}^2 + O\big(m_{\rm toy}^4\big), \tag{4.16}$$

with Λ^2 given by (3.9).

Comparing relations (4.12) and (4.15), we observe that $m_{\rm O(6)}$ and $m_{\rm toy}$ have the same leading asymptotics while subleading $1/g$ corrections to the two scales have different transcendentality. Namely, the perturbative coefficients in $m_{\rm toy}$ are rational numbers while for $m_{\rm O(6)}$ their transcendentality increases with order in $1/g$. We recall that we already encountered the same property for the cusp anomalous dimension (equations (2.59) and (2.63)). There, we have observed that the two expressions (2.59) and (2.63) coincide upon the substitution (2.64). Performing the same substitution in (4.12) we find that, remarkably enough, the two expressions for the mass gap indeed coincide up to an overall normalization factor:

$$m_{\rm O(6)} \stackrel{\text{equation (2.64)}}{=} \frac{\pi}{2\sqrt{2}} m_{\rm toy}. \tag{4.17}$$

The expressions for the cusp anomalous dimension (4.11) and for the mass scale (4.12) can be further simplified if one redefines the coupling constant as

$$g' = g - c_1, \qquad c_1 = \frac{3\ln 2}{4\pi}, \tag{4.18}$$

and re-expands both quantities in $1/g'$. As was observed in [25], such redefinition allows one to eliminate 'ln 2' terms in perturbative expansion of the cusp anomalous dimension. Repeating the same analysis for (4.11), we find that the same is also true for nonperturbative corrections:

$$\Gamma_{\rm cusp}\,(g + c_1) = 2g\left[1 - \frac{\mathrm{K}}{(4\pi g)^2} + \cdots\right] - \frac{\Lambda^2}{2\sqrt{2}\pi}\left[1 + \frac{3}{16\pi g} + \cdots\right] + O(\Lambda^4), \tag{4.19}$$

with Λ^2 defined in (3.9). In a similar manner, the expression for the mass scale (4.12) takes the form

$$\begin{aligned}[m_{\rm O(6)}(g + c_1)]^2 = \frac{2\Lambda^2}{\pi\sigma}\Bigg[&\left(1 + \frac{3}{16\pi g} + \frac{16\mathrm{K} - 54}{512(\pi g)^2} + \cdots\right) \\ &- \frac{\Lambda^2}{8\sqrt{2}\pi g}\left(1 - \frac{3}{8\pi g} + \cdots\right) + O(\Lambda^4)\Bigg].\end{aligned} \tag{4.20}$$

Comparing relations (4.19) and (4.20), we immediately recognize that, within an accuracy of the expressions obtained, the nonperturbative $O(\Lambda^2)$ correction to the cusp anomalous dimension is given by $m_{\rm O(6)}^2$:

$$\delta\Gamma_{\rm cusp} = -\frac{\sigma}{4\sqrt{2}} m_{\rm O(6)}^2 + O\big(m_{\rm O(6)}^4\big). \tag{4.21}$$

It is worth mentioning that, upon identification of the scales (4.17), this relation coincides with (4.16).

We will show in the following subsection that relation (4.21) holds at strong coupling to all orders in $1/g$.

4.4. Relation between the cusp anomalous dimension and mass gap

We demonstrated that the strong coupling expansion of the cusp anomalous dimension has form (3.24) with the leading nonperturbative correction given to the first few orders in $1/g$ expansion by the mass scale of the O(6) model, $m^2_{\rm cusp} = m^2_{\rm O(6)}$. Let us show that this relation is in fact exact at strong coupling.

According to (4.10), the leading nonperturbative correction to the cusp anomalous dimension is given by

$$\delta\Gamma_{\rm cusp} = -4g\delta f_1(0), \tag{4.22}$$

with $\delta f_1(0)$ denoting the $O(\Lambda^2)$ correction to the function $f_1(t=0)$ (equation (4.6)). We recall that this function verifies the quantization conditions (2.45). As was explained in section 3.3, the leading $O(\Lambda^2)$ corrections to solutions of (2.45) originate from subleading, exponentially suppressed terms in the strong coupling expansion of the functions $V_0(-\pi g)$ and $V_1(-\pi g)$ that we shall denote as $\delta V_0(-\pi g)$ and $\delta V_1(-\pi g)$, respectively. Using the identities (D.14) and (D.16), we find

$$\delta V_0(-\pi g) = \sigma\frac{2\sqrt{2}}{\pi}\,\mathrm{e}^{-\pi g}U_0^-(\pi g), \qquad \delta V_1(-\pi g) = \sigma\frac{2\sqrt{2}}{\pi}\,\mathrm{e}^{-\pi g}U_1^-(\pi g), \tag{4.23}$$

where the functions $U_0^-(\pi g)$ and $U_1^-(\pi g)$ are defined in (D.7). Then, we split the functions $f_0(t)$ and $f_1(t)$ entering the quantization conditions (2.45) into perturbative and nonperturbative parts according to (4.6) and compare exponentially small terms on both sides of (2.45) to get

$$\delta f_0(t_\ell)V_0(t_\ell) + \delta f_1(t_\ell)V_1(t_\ell) = -m'\delta_{\ell,0}, \tag{4.24}$$

where $t_\ell = 4\pi g\left(\ell - \frac{1}{4}\right)$ and the notation was introduced for

$$m' = f_0(-\pi g)\delta V_0(-\pi g) + f_1(-\pi g)\delta V_1(-\pi g). \tag{4.25}$$

Taking into account relations (4.23) and comparing the resulting expression for m' with (4.4), we find that

$$m' = -\frac{\sigma}{8g}m_{\rm O(6)}, \tag{4.26}$$

with $m_{\rm O(6)}$ being the mass scale (4.4).

To compute nonperturbative $O(\Lambda^2)$ correction to the cusp anomalous dimension, we have to solve the system of relations (4.24), determine the function $\delta f_1(t)$ and, then, apply (4.22). We will show in this subsection that the result reads as

$$\delta f_1(0) = -\frac{\sqrt{2}}{4}m' m_{\rm O(6)} = \sigma\frac{\sqrt{2}}{32g}m^2_{\rm O(6)}, \tag{4.27}$$

to all orders in strong coupling expansion. Together with (4.22), this leads to the desired expression (4.21) for leading nonperturbative correction to the cusp anomalous dimension.

To begin with, let us introduce a new function analogous to (2.39):

$$\delta\Gamma(\mathrm{i}t) = \delta f_0(t)V_0(t) + \delta f_1(t)V_1(t). \tag{4.28}$$

Here $\delta f_0(t)$ and $\delta f_1(t)$ are given by the same expressions as before, equation (2.40), with the only difference that the coefficients $c_\pm(n,g)$ are replaced in (2.40) by their leading nonperturbative correction $\delta c_\pm(n,g) = O(\Lambda^2)$ and relation (3.4) is taken into account. This implies that various relations for $\Gamma(\mathrm{i}t)$ can be immediately translated into those for the function $\delta\Gamma(\mathrm{i}t)$. In particular, for $t=0$ we find from (2.40) that $\delta f_0(0) = 0$ for arbitrary coupling, leading to

$$\delta\Gamma(0) = 2\delta f_1(0). \tag{4.29}$$

In addition, we recall that, for arbitrary $c_\pm(n,g)$, function (2.39) satisfies the inhomogeneous integral equation (2.7). In other words, the $c_\pm(n,g)$-dependent terms in the expression for the function $\Gamma(\mathrm{i}t)$ are zero modes for the integral equation (2.7). Since function (4.28) is just given by the sum of such terms, it automatically satisfies the homogenous equation

$$\int_0^\infty \mathrm{d}t[\mathrm{e}^{\mathrm{i}tu}\delta\Gamma_-(t) - \mathrm{e}^{-\mathrm{i}tu}\delta\Gamma_+(t)] = 0, \qquad (-1 \leqslant u \leqslant 1), \tag{4.30}$$

where $\delta\Gamma(t) = \delta\Gamma_+(t) + \mathrm{i}\delta\Gamma_-(t)$ and $\delta\Gamma_\pm(-t) = \pm\delta\Gamma_\pm(t)$.

As before, in order to construct a solution to (4.30), we have to specify additional conditions for $\delta\Gamma(t)$. Since the substitution $c_\pm(n,g) \to \delta c_\pm(n,g)$ does not affect the analytical properties of functions (2.40), function (4.28) shares with $\Gamma(\mathrm{i}t)$ an infinite set of simple poles located at the same position (2.14):

$$\delta\Gamma(\mathrm{i}t) \sim \frac{1}{t - 4\pi g\ell}, \qquad (\ell \in \mathbb{Z}/0). \tag{4.31}$$

In addition, we deduce from (4.24) that it also satisfies the relation (with $x_\ell = \ell - \frac{1}{4}$)

$$\delta\Gamma(4\pi\mathrm{i}gx_\ell) = -m'\delta_{\ell,0}, \qquad (\ell \in \mathbb{Z}), \tag{4.32}$$

and, therefore, has an infinite number of zeros. An important difference with $\Gamma(\mathrm{i}t)$ is that $\delta\Gamma(\mathrm{i}t)$ does not vanish at $t = -\pi g$ and its value is fixed by the parameter m' defined in (4.26).

Keeping in mind the similarity between the functions $\Gamma(\mathrm{i}t)$ and $\delta\Gamma(\mathrm{i}t)$, we follow (2.13) and define a new function

$$\delta\gamma(\mathrm{i}t) = \frac{\sin(t/4g)}{\sqrt{2}\sin(t/4g + \pi/4)}\delta\Gamma(\mathrm{i}t). \tag{4.33}$$

As before, the poles and zeros of $\widehat{\Gamma}(\mathrm{i}t)$ are compensated by the ratio of sinus functions. However, in distinction with $\gamma(\mathrm{i}t)$ and in virtue of $\delta\Gamma(-\pi\mathrm{i}g) = -m'$, the function $\delta\gamma(\mathrm{i}t)$ has a single pole at $t = -\pi g$ with the residue equal to $2gm'$. For $t \to 0$, we find from (4.33) that $\delta\gamma(\mathrm{i}t)$ vanishes as

$$\delta\gamma(\mathrm{i}t) = \frac{t}{4g}\delta\Gamma(0) + O(t^2) = \frac{t}{2g}\delta f_1(0) + O(t^2), \tag{4.34}$$

where in the second relation we applied (4.29). It is convenient to split the function $\delta\gamma(t)$ into the sum of two terms of a definite parity, $\delta\gamma(t) = \delta\gamma_+(t) + \mathrm{i}\delta\gamma_-(t)$ with $\delta\gamma_\pm(-t) = \pm\delta\gamma_\pm(t)$. Then, combining together (4.30) and (4.33) we obtain that the functions $\delta\gamma_\pm(t)$ satisfy the infinite system of homogenous equations (for $n \geqslant 1$):

$$\begin{aligned}
&\int_0^\infty \frac{\mathrm{d}t}{t} J_{2n-1}(t)\left[\frac{\delta\gamma_-(t)}{1-\mathrm{e}^{-t/2g}} + \frac{\delta\gamma_+(t)}{\mathrm{e}^{t/2g}-1}\right] = 0,\\
&\int_0^\infty \frac{\mathrm{d}t}{t} J_{2n}(t)\left[\frac{\delta\gamma_+(t)}{1-\mathrm{e}^{-t/2g}} - \frac{\delta\gamma_-(t)}{\mathrm{e}^{t/2g}-1}\right] = 0.
\end{aligned} \tag{4.35}$$

By construction, the solution to this system $\delta\gamma(t)$ should vanish at $t = 0$ and have a simple pole at $t = -\mathrm{i}\pi g$.

As was already explained, the functions $\delta\Gamma_\pm(t)$ satisfy the same integral equation (4.30) as the function $\Gamma_\pm(t)$ up to an inhomogeneous term on the right-hand side of (2.7). Therefore, it should not be surprising that the system (4.35) coincides with relations (2.3) after one neglects the inhomogeneous term on the right-hand side of (2.3). As we show in appendix C, this fact allows us to derive Wronskian-like relations between the functions $\delta\gamma(t)$ and $\gamma(t)$. These relations turn out to be powerful enough to determine the small t asymptotics of the function $\delta\gamma(t)$ at small t in terms of $\gamma(t)$, or equivalently $\Gamma(t)$. In this way, we obtain (see appendix C for more detail)

$$\delta\gamma(\mathrm{i}t) = -m't\left[\frac{2}{\pi^2 g}\,\mathrm{e}^{-\pi g} - \frac{\sqrt{2}}{\pi}\,\mathrm{e}^{-\pi g}\,\mathrm{Re}\int_0^\infty \frac{\mathrm{d}t'}{t'+\mathrm{i}\pi g}\,\mathrm{e}^{\mathrm{i}(t'-\pi/4)}\Gamma(t')\right] + O(t^2). \tag{4.36}$$

Comparing this relation with (2.9), we realize that the expression inside the square brackets is proportional to the mass scale $m_{\mathrm{O}(6)}$ leading to

$$\delta\gamma(\mathrm{i}t) = -m'm_{\mathrm{O}(6)}\frac{t\sqrt{2}}{8g} + O(t^2). \tag{4.37}$$

Matching this relation to (4.34), we obtain the desired expression for $\delta f_1(0)$ (equation (4.27)). Then, we substitute it into (4.22) and compute the leading nonperturbative correction to the cusp anomalous dimension, equation (4.21), leading to

$$m_{\mathrm{cusp}}(g) = m_{\mathrm{O}(6)}(g). \tag{4.38}$$

Thus, we demonstrated in this section that nonperturbative, exponentially small corrections to the cusp anomalous dimensions at strong coupling are determined to all orders in $1/g$ by the mass gap of the two-dimensional bosonic O(6) model embedded into the $\mathrm{AdS}_5 \times \mathrm{S}^5$ sigma model.

5. Conclusions

In this paper, we have studied the anomalous dimensions of Wilson operators in the $SL(2)$ sector of planar $\mathcal{N} = 4$ SYM theory in the double scaling limit when the Lorentz spin of the operators grows exponentially with their twist. In this limit, the asymptotic behavior of the anomalous dimensions is determined by the cusp anomalous dimension $\Gamma_{\mathrm{cusp}}(g)$ and the scaling function $\epsilon(g, j)$. We found that at strong coupling, both functions receive exponentially small corrections which are parameterized by the same nonperturbative scale. It is remarkable that this scale appears on both sides of the AdS/CFT correspondence. In string theory, it emerges as the mass gap of the two-dimensional bosonic O(6) sigma model which describes the effective dynamics of massless excitations for a folded spinning string in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ sigma model [6].

The dependence of $\Gamma_{\mathrm{cusp}}(g)$ and $\epsilon(g, j)$ on the coupling constant is governed by integral BES/FRS equations which follow from the conjectured all-loop integrability of the dilatation operator of the $\mathcal{N} = 4$ model. At weak coupling, their solutions agree with the results of explicit perturbative calculations. At strong coupling, a systematic expansion of the cusp anomalous dimension in powers of $1/g$ was derived in [25]. In agreement with the AdS/CFT correspondence, the first few terms of this expansion coincide with the energy of the semiclassically quantized folded spinning strings. However, the expansion coefficients grow factorially at higher orders and, as a consequence, the 'perturbative' $1/g$ expansion of the cusp anomalous dimension suffers from Borel singularities which induce exponentially small corrections to $\Gamma_{\mathrm{cusp}}(g)$. To identify such nonperturbative corrections, we revisited the BES equation and constructed the exact solution for the cusp anomalous dimension valid for an arbitrary coupling constant.

At strong coupling, we found that the expression obtained for $\Gamma_{\mathrm{cusp}}(g)$ depends on a new scale $m_{\mathrm{cusp}}(g)$ which is exponentially small as $g \to \infty$. Nonperturbative corrections to $\Gamma_{\mathrm{cusp}}(g)$ at strong coupling run in even powers of this scale, and the coefficients of this expansion depend on the prescription employed to regularize Borel singularities in perturbative $1/g$ series. It is only the sum of perturbative and nonperturbative contributions which is independent of the choice of the prescription. For the scaling function $\epsilon(g, j)$, the defining integral FRS equation can be brought to the form of the thermodynamical Bethe ansatz equations for the energy density of the ground state of the O(6) model. As a consequence, nonperturbative contribution to $\epsilon(g, j)$ at strong coupling is described by the mass scale of this model $m_{O(6)}(g)$. We have shown that the two scales coincide, $m_{\rm cusp}(g) = m_{O(6)}(g)$, and, therefore, nonperturbative contributions to $\Gamma_{\rm cusp}(g)$ and $\epsilon(g, j)$ are governed by the same scale $m_{O(6)}(g)$.

This result agrees with the proposal by Alday–Maldacena that, in string theory, the leading nonperturbative corrections to the cusp anomalous dimension coincide with those to the vacuum energy density of the two-dimensional bosonic O(6) model embedded into the $AdS_5 \times S^5$ sigma model. These models have different properties: the former model has asymptotic freedom at short distances and develops the mass gap in the infrared while the latter model is conformal. The O(6) model only describes an effective dynamics of massless modes of $AdS_5 \times S^5$ and the mass of massive excitations $\mu \sim 1$ defines a ultraviolet (UV) cutoff for this model. The coupling constants in the two models are related to each other as $\bar{g}^2(\mu) = 1/(2g)$. The vacuum energy density in the O(6) model and more generally in the O(n) model is an ultraviolet-divergent quantity. It also depends on the mass scale of the model and has the following form:

$$\epsilon_{\rm vac} = \mu^2 \epsilon(\bar{g}^2) + \kappa m_{O(n)}^2 + O\left(m_{O(n)}^4 / \mu^2\right). \tag{5.1}$$

Here μ^2 is a UV cutoff, $\epsilon(\bar{g}^2)$ stands for the perturbative series in $\bar{g}^2$ and the mass gap $m_{O(n)}^2$ is

$$m_{O(n)}^2 = c\mu^2 \, e^{-\frac{1}{\beta_0 \bar{g}^2}} \, \bar{g}^{-2\beta_1/\beta_0^2} [1 + O(\bar{g}^2)], \tag{5.2}$$

where β_0 and β_1 are the beta-function coefficients for the O(n) model and the normalization factor c ensures independence of $m_{O(n)}$ on the renormalization scheme. For $n = 6$, relation (5.2) coincides with (1.3) and the expression for the vacuum energy density (5.1) should be compared with (1.2).

The two terms on the right-hand side of (5.1) describe perturbative and nonperturbative corrections to $\epsilon_{\rm vac}$. For $n \to \infty$, each of them is well defined separately and can be computed exactly [48, 49]. For n finite, including $n = 6$, the function $\epsilon(\bar{g}^2)$ is given in a generic renormalization scheme by a non-Borel summable series and, therefore, is not well defined. In a close analogy with (1.2), the coefficient κ in front of $m_{O(n)}^2$ on the right-hand side of (5.1) depends on the regularization of Borel singularities in perturbative series for $\epsilon(\bar{g}^2)$. Note that $\epsilon_{\rm vac}$ is related to the vacuum expectation value of the trace of the tensor energy–momentum in the two-dimensional O(n) sigma model [49]. The AdS/CFT correspondence implies that for $n = 6$, the same quantity defines the nonperturbative correction to the cusp anomalous dimension (1.2). It would be interesting to obtain its dual representation (if any) in terms of certain operators in four-dimensional $\mathcal{N} = 4$ SYM theory. Finally, one may wonder whether it is possible to identify a restricted class of Feynman diagrams in $\mathcal{N} = 4$ theory whose resummation could produce contribution to the cusp anomalous dimension exponentially small as $g \to \infty$. As a relevant example, we would like to mention that exponentially suppressed corrections were obtained in [50] from the exact resummation of ladder diagrams in four-dimensional massless $g\phi^3$ theory.

Acknowledgments

We would like to thank Janek Kotański for collaboration at the early stages of this work and Zoltan Bajnok, Janos Balog, David Broadhurst, François David, Alexander Gorsky, Juan Maldacena, Laszlo Palla, Mikhail Shifman, Arkady Tseytlin and Dima Volin for interesting discussions. This work was supported in part by the French Agence Nationale de la Recherche under grant ANR-06-BLAN-0142.

Appendix A. Weak coupling expansion

In this appendix, we work out the first few terms of the weak coupling expansion of the coefficient $c(g)$ entering (2.47) and show that they vanish in agreement with (3.4). To this end, we will not attempt at solving the quantization conditions (2.45) at weak coupling but will use instead the fact that the BES equation can be solved by iteration of the inhomogeneous term.

The system of integral equation (2.3) can be easily solved at weak coupling by looking for its solutions $\gamma_\pm(t)$ in the form of the Bessel series (2.10) and expanding the coefficients γ_{2k} and γ_{2k-1} in powers of the coupling constant. For $g \to 0$, it follows from (2.3) and from orthogonality conditions for the Bessel functions that $\gamma_-(t) = J_1(t) + \cdots$ and $\gamma_+(t) = 0 + \cdots$ with the ellipses denoting subleading terms. To determine such terms, it is convenient to change the integration variable in (2.3) as $t \to tg$. Then, taking into account the relations $J_k(-gt) = (-1)^k J_k(gt)$ we observe that the resulting equations are invariant under substitution $g \to -g$ provided that the functions $\gamma_\pm(gt)$ change sign under this transformation. Since $\gamma_\pm(-t) = \pm\gamma_\pm(t)$, this implies that the coefficients $\gamma_{2n-1}(g)$ and $\gamma_{2n}(g)$ entering (2.10) have a definite parity as functions of the coupling constant

$$\gamma_{2n-1}(-g) = \gamma_{2n-1}(g), \qquad \gamma_{2n}(-g) = -\gamma_{2n}(g), \tag{A.1}$$

and, therefore, their weak coupling expansion runs in even and odd powers of g, respectively. Expanding both sides of (2.3) at weak coupling and comparing the coefficients in front of powers of g, we find

$$\begin{aligned}
\gamma_1 &= \frac{1}{2} - \frac{\pi^2}{6}g^2 + \frac{11\pi^4}{90}g^4 - \left(\frac{73\pi^6}{630} + 4\zeta_3^2\right)g^6 + O(g^8),\\
\gamma_2 &= \zeta_3 g^3 - \left(\frac{\pi^2}{3}\zeta_3 + 10\zeta_5\right)g^5 + \left(\frac{8\pi^4}{45}\zeta_3 + \frac{10\pi^2}{3}\zeta_5 + 105\zeta_7\right)g^7 + O(g^9),\\
\gamma_3 &= -\frac{\pi^4}{90}g^4 + \frac{37\pi^6}{1890}g^6 + O(g^8), \qquad \gamma_4 = \zeta_5 g^5 - \left(\frac{\pi^2}{3}\zeta_5 + 21\zeta_7\right)g^7 + O(g^9),\\
\gamma_5 &= -\frac{\pi^6}{945}g^6 + O(g^8), \qquad \gamma_6 = \zeta_7 g^7 + O(g^9).
\end{aligned} \tag{A.2}$$

We verify with the help of (2.12) that the expression for the cusp anomalous dimension

$$\Gamma_{\text{cusp}}(g) = 8g^2\gamma_1(g) = 4g^2 - \frac{4\pi^2}{3}g^4 + \frac{44\pi^4}{45}g^6 - \left(\frac{292\pi^6}{315} + 32\zeta_3^2\right)g^8 + O(g^{10}) \tag{A.3}$$

agrees with the known four-loop result in planar $\mathcal{N} = 4$ SYM theory [20].

In our approach, the cusp anomalous dimension is given for an arbitrary value of the coupling constant by expression (2.47) which involves the functions $c(g)$ and $c_\pm(n, g)$. According to (2.29), the latter functions are related to the functions $\gamma(t) = \gamma_+(t) + \mathrm{i}\gamma_-(t)$ evaluated at $t = 4\pi \mathrm{i} gn$:

$$\begin{aligned}
c_+(n, g) &= -4g\, \mathrm{e}^{-4\pi gn}[\gamma_+(4\pi \mathrm{i} gn) + \mathrm{i}\gamma_-(4\pi \mathrm{i} gn)],\\
c_-(n, g) &= \quad 4g\mathrm{e}^{-4\pi gn}[\gamma_+(4\pi \mathrm{i} gn) - \mathrm{i}\gamma_-(4\pi \mathrm{i} gn)].
\end{aligned} \tag{A.4}$$

At strong coupling, we determined $c_\pm(n, g)$ by solving the quantization conditions (3.6). At weak coupling, we can compute $c_\pm(n, g)$ from (A.4) by replacing $\gamma_\pm(t)$ with their Bessel series (2.10) and making use of the expressions obtained for the expansion coefficients (A.2).

The remaining function $c(g)$ can be found from comparison of two different representations for the cusp anomalous dimension (equations (2.47) and (2.12)):

$$c(g) = -\frac{1}{2} + 2g\gamma_1(g) + \sum_{n\geqslant 1}\big[c_-(n,g)U_0^-(4\pi ng) + c_+(n,g)U_0^+(4\pi ng)\big]. \tag{A.5}$$

Taking into account relations (A.4) and (2.10), we find

$$c(g) = -\frac{1}{2} + 2g\gamma_1(g) - \sum_{k\geqslant 1}(-1)^k[(2k-1)\gamma_{2k-1}(g)f_{2k-1}(g) + (2k)\gamma_{2k}(g)f_{2k}(g)], \tag{A.6}$$

where the coefficients γ_k are given by (A.2) and the notation was introduced for the functions

$$f_k(g) = 8g\sum_{n\geqslant 1}\big[U_0^+(4\pi gn) - (-1)^k U_0^-(4\pi gn)\big]I_k(4\pi gn)\,\mathrm{e}^{-4\pi gn}. \tag{A.7}$$

Here, $I_k(x)$ is the modified Bessel function [47] and the functions $U_0^{\pm}(x)$ are defined in (D.7). At weak coupling, the sum over n can be evaluated with the help of the Euler–Maclaurin summation formula. Going through lengthy calculation, we find

$$\begin{aligned}
&f_1 = 1 - 2g + \frac{\pi^2}{3}g^2 + 2\zeta_3 g^3 - \frac{\pi^4}{6}g^4 - 23\zeta_5 g^5 + \frac{17\pi^6}{108}g^6 + \frac{1107}{4}\zeta_7 g^7 + O(g^8),\\
&f_2 = -\frac{1}{2} + 2\zeta_3 g^3 - \frac{\pi^4}{30}g^4 + O(g^5), \qquad f_3 = \frac{1}{2} + O(g^4),\\
&f_4 = -\frac{3}{8} + O(g^5), \qquad f_5 = \frac{3}{8} + O(g^6), \qquad f_6 = -\frac{5}{16} + O(g^7).
\end{aligned} \tag{A.8}$$

In this way, we obtain from (A.6)

$$c(g) = -\tfrac{1}{2} + (f_1 + 2g)\gamma_1 + 2f_2\gamma_2 - 3f_3\gamma_3 - 4f_4\gamma_4 + 5f_5\gamma_5 + 6f_6\gamma_6 + \cdots = O(g^8). \tag{A.9}$$

Thus, in agreement with (3.4), the function $c(g)$ vanishes at weak coupling. As was shown in section 3.1, the relation $c(g) = 0$ holds for arbitrary coupling.

Appendix B. Constructing general solution

By construction, the function $\Gamma(t) = \Gamma_+(t) + \mathrm{i}\Gamma_-(t)$ defined as the exact solution to the integral equation (2.7) is given by the Fourier integral

$$\Gamma(t) = \int_{-\infty}^{\infty}\mathrm{d}k\,\mathrm{e}^{-\mathrm{i}kt}\widetilde{\Gamma}(k), \tag{B.1}$$

with the function $\widetilde{\Gamma}(k)$ having a different form for $k^2 \leqslant 1$ and $k^2 > 1$:

- For $-\infty < k < -1$:

$$\widetilde{\Gamma}(k) = \sum_{n\geqslant 1} c_-(n,g)\,\mathrm{e}^{-4\pi ng(-k-1)}, \tag{B.2}$$

- For $1 < k < \infty$:

$$\widetilde{\Gamma}(k) = \sum_{n\geqslant 1} c_+(n,g)\,\mathrm{e}^{-4\pi ng(k-1)}, \tag{B.3}$$

- For $-1 \leqslant k \leqslant 1$:

$$\widetilde{\Gamma}(k) = -\frac{\sqrt{2}}{\pi}\left(\frac{1+k}{1-k}\right)^{1/4}\left[1 + \frac{c(g)}{1+k} + \frac{1}{2}\left(\int_{-\infty}^{-1} + \int_1^{\infty}\right)\frac{\mathrm{d}p\,\widetilde{\Gamma}(p)}{p-k}\left(\frac{p-1}{p+1}\right)^{1/4}\right], \tag{B.4}$$

where $\widetilde{\Gamma}(p)$ inside the integral is replaced by (B.2) and (B.3).

Let us split the integral in (B.1) into three terms as in (2.38) and evaluate them one after another. Integration over $k^2 > 1$ can be done immediately while the integral over $-1 \leqslant k \leqslant 1$ can be expressed in terms of special functions:

$$\Gamma(t) = \sum_{n\geqslant 1} c_+(n,g)\left[\frac{\mathrm{e}^{-\mathrm{i}t}}{4\pi ng+\mathrm{i}t} - V_+(-\mathrm{i}t, 4\pi ng)\right] + \sum_{n\geqslant 1} c_-(n,g)\left[\frac{\mathrm{e}^{\mathrm{i}t}}{4\pi ng-\mathrm{i}t} + V_-(\mathrm{i}t, 4\pi ng)\right] - V_0(-\mathrm{i}t) - c(g)V_1(-\mathrm{i}t), \tag{B.5}$$

where the notation was introduced for the functions (with $n = 0, 1$)

$$\begin{aligned} V_\pm(x,y) &= \frac{1}{\sqrt{2}\pi}\int_{-1}^{1} \mathrm{d}k\ \mathrm{e}^{\pm xk}\int_1^\infty \frac{\mathrm{d}p\mathrm{e}^{-y(p-1)}}{p-k}\left(\frac{1+k}{1-k}\,\frac{p-1}{p+1}\right)^{\pm 1/4},\\ V_n(x) &= \frac{\sqrt{2}}{\pi}\int_{-1}^{1}\frac{\mathrm{d}k\ \mathrm{e}^{xk}}{(k+1)^n}\left(\frac{1+k}{1-k}\right)^{1/4},\\ U_n^\pm(y) &= \frac{1}{2}\int_1^\infty \frac{\mathrm{d}p\ \mathrm{e}^{-y(p-1)}}{(p\mp 1)^n}\left(\frac{p+1}{p-1}\right)^{\mp 1/4}. \end{aligned} \tag{B.6}$$

The reason why we also introduced $U_n^\pm(y)$ is that the functions $V_\pm(x,y)$ can be further simplified with the help of master identities (we shall return to them in a moment):

$$\begin{aligned} (x+y)V_-(x,y) &= xV_0(x)U_1^-(y) + yV_1(x)U_0^-(y) - \mathrm{e}^{-x},\\ (x-y)V_+(x,y) &= xV_0(x)U_1^+(y) + yV_1(x)U_0^+(y) - \mathrm{e}^{x}. \end{aligned} \tag{B.7}$$

Combining together (B.7) and (B.5), we arrive at the following expression for the function $\Gamma(\mathrm{i}t)$:

$$\Gamma(\mathrm{i}t) = -V_0(t) - c(g)V_1(t) + \sum_{n\geqslant 1} c_+(n,g)\left[\frac{4\pi ngV_1(t)U_0^+(4\pi ng) + tV_0(t)U_1^+(4\pi ng)}{4\pi ng - t}\right] + \sum_{n\geqslant 1} c_-(n,g)\left[\frac{4\pi ngV_1(t)U_0^-(4\pi ng) + tV_0(t)U_1^-(4\pi ng)}{4\pi ng + t}\right], \tag{B.8}$$

which leads to (2.39).

We show in appendix D that the functions $V_{0,1}(t)$ and $U_{0,1}^\pm(4\pi ng)$ can be expressed in terms of the Whittaker functions of the first and second kinds, respectively. As follows from their integral representation, $V_0(t)$ and $V_1(t)$ are holomorphic functions of t. As a result, $\Gamma(\mathrm{i}t)$ is a meromorphic function of t with a (infinite) set of poles located at $t = \pm 4\pi ng$ with n being a positive integer.

Let us now prove the master identities (B.7). We start with the second relation in (B.7) and make use of (B.6) to rewrite the expression on the left-hand side of (B.7) as

$$(x-y)V_+(x,y)\,\mathrm{e}^{-y} = (x-y)\int_0^\infty \mathrm{d}s\,V_0(x+s)U_0^+(y+s)\,\mathrm{e}^{-y-s}. \tag{B.9}$$

Let us introduce two auxiliary functions

$$\begin{aligned} z_1(x) &= V_1(x), & z_1(x) + z_1'(x) &= V_0(x),\\ z_2(x) &= \mathrm{e}^{-x}U_1^+(x), & z_2(x) + z_2'(x) &= -\mathrm{e}^{-x}U_0^+(x), \end{aligned} \tag{B.10}$$

with $V_n(x)$ and $U_n^+(x)$ being given by (B.6). They satisfy the second-order differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}(xz_i'(x)) = \left(x - \frac{1}{2}\right)z_i(x). \tag{B.11}$$

Applying this relation, it is straightforward to verify the following identity:

$$
\begin{aligned}
-(x-y)&[z_1(x+s)+z_1'(x+s)][z_2(y+s)+z_2'(y+s)] \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\{(y+s)[z_2(y+s)+z_2'(y+s)]z_1(x+s)\} \\
&\quad - \frac{\mathrm{d}}{\mathrm{d}s}\{(x+s)[z_1(x+s)+z_1'(x+s)]z_2(y+s)\}.
\end{aligned} \tag{B.12}
$$

It is easy to see that the expression on the left-hand side coincides with the integrand in (B.9). Therefore, integrating both sides of (B.12) over $0 \leqslant s < \infty$, we obtain

$$
\begin{aligned}
(x-y)V_+(x,y) &= -\mathrm{e}^{-s}\left[(x+s)V_0(x+s)U_1^+(y+s)+(y+s)V_1(x+s)U_0^+(y+s)\right]\Big|_{s=0}^{s=\infty} \\
&= -\mathrm{e}^{x} + xV_0(x)U_1^+(y) + yV_1(x)U_0^+(y),
\end{aligned} \tag{B.13}
$$

where in the second relation we took into account the asymptotic behavior of functions (B.6) (see equations (D.10) and (D.12)), $V_n(s) \sim \mathrm{e}^s s^{-3/4}$ and $U_n^+(s) \sim s^{n-5/4}$ as $s \to \infty$.

The derivation of the first relation in (B.7) goes along the same lines.

Appendix C. Wronskian-like relations

In this appendix, we present a detailed derivation of relation (4.36) which determines the small t expansion of the function $\delta\gamma(t)$. This function satisfies the infinite system of integral equations (4.35). In addition, it should vanish at the origin, $t = 0$, and have a simple pole at $t = -\mathrm{i}\pi g$ with the residue $2\mathrm{i}gm'$ (see equation (4.33)). To fulfill these requirements, we split $\delta\gamma(\mathrm{i}t)$ into the sum of two functions:

$$
\delta\gamma(\mathrm{i}t) = \widehat{\gamma}(\mathrm{i}t) - \frac{2m'}{\pi}\frac{t}{t+\pi g}, \tag{C.1}
$$

where, by construction, $\widehat{\gamma}(\mathrm{i}t)$ is an entire function vanishing at $t = 0$ and its Fourier transform has a support on the interval $[-1, 1]$. Similar to (2.2), we decompose $\delta\gamma(t)$ and $\widehat{\gamma}(t)$ into the sum of two functions with a definite parity

$$
\begin{aligned}
\delta\gamma_+(t) &= \widehat{\gamma}_+(t) - \frac{2m'}{\pi}\frac{t^2}{t^2+(\pi g)^2}, \\
\delta\gamma_-(t) &= \widehat{\gamma}_-(t) + \frac{2gm't}{t^2+\pi^2 g^2}.
\end{aligned} \tag{C.2}
$$

Then, we substitute these relations into (4.35) and obtain the system of inhomogeneous integral equations for the functions $\widehat{\gamma}_\pm(t)$:

$$
\begin{aligned}
&\int_0^\infty \frac{\mathrm{d}t}{t} J_{2n-1}(t)\left[\frac{\widehat{\gamma}_-(t)}{1-\mathrm{e}^{-t/2g}} + \frac{\widehat{\gamma}_+(t)}{\mathrm{e}^{t/2g}-1}\right] = h_{2n-1}(g), \\
&\int_0^\infty \frac{\mathrm{d}t}{t} J_{2n}(t)\left[\frac{\widehat{\gamma}_+(t)}{1-e^{-t/2g}} - \frac{\widehat{\gamma}_-(t)}{e^{t/2g}-1}\right] = h_{2n}(g),
\end{aligned} \tag{C.3}
$$

with inhomogeneous terms being given by

$$
\begin{aligned}
h_{2n-1} &= \frac{2m'}{\pi}\int_0^\infty \frac{\mathrm{d}t\, J_{2n-1}(t)}{t^2+(\pi g)^2}\left[\frac{t}{\mathrm{e}^{t/(2g)}-1} - \frac{\pi g}{1-\mathrm{e}^{-t/(2g)}}\right], \\
h_{2n} &= \frac{2m'}{\pi}\int_0^\infty \frac{\mathrm{d}t\, J_{2n}(t)}{t^2+(\pi g)^2}\left[\frac{\pi g}{\mathrm{e}^{t/(2g)}-1} + \frac{t}{1-\mathrm{e}^{-t/(2g)}}\right].
\end{aligned} \tag{C.4}
$$

Comparing these relations with (C.3) we observe that they only differ by the form of inhomogeneous terms and can be obtained from one another through the substitution

$$
\widehat{\gamma}_\pm(t) \to \gamma_\pm(t), \qquad h_{2n-1} \to \tfrac{1}{2}\delta_{n,1}, \qquad h_{2n} \to 0. \tag{C.5}
$$

In a close analogy with (2.10), we look for a solution to (C.3) in the form of Bessel series:

$$\begin{aligned}\widehat{\gamma}_-(t) &= 2\sum_{n\geqslant 1}(2n-1)J_{2n-1}(t)\widehat{\gamma}_{2n-1}(g),\\ \widehat{\gamma}_+(t) &= 2\sum_{n\geqslant 1}(2n)J_{2n}(t)\widehat{\gamma}_{2n}(g).\end{aligned} \tag{C.6}$$

For small t, we have $\widehat{\gamma}_-(t) = t\widehat{\gamma}_1 + O(t^2)$ and $\widehat{\gamma}_+(t) = O(t^2)$. Then it follows from (C.1) that

$$\delta\gamma(t) = \mathrm{i}\widehat{\gamma}_-(t) + \frac{2\mathrm{i}m'}{\pi^2 g}t + O(t^2) = \mathrm{i}t\left(\widehat{\gamma}_1 + \frac{2m'}{\pi^2 g}\right) + O(t^2), \tag{C.7}$$

so that the leading asymptotics is controlled by the coefficient $\widehat{\gamma}_1$.

Let us multiply both sides of the first relation in (C.3) by $(2n-1)\gamma_{2n-1}$ and sum both sides over $n \geqslant 1$ with the help of (2.10). In a similar manner, we multiply the second relation in (C.3) by $(2n)\gamma_{2n}$ and follow the same steps. Then, we subtract the second relation from the first one and obtain

$$\begin{aligned}\int_0^\infty \frac{\mathrm{d}t}{t}\left[\frac{\gamma_-(t)\widehat{\gamma}_-(t) - \gamma_+(t)\widehat{\gamma}_+(t)}{1-\mathrm{e}^{-t/2g}} + \frac{\gamma_-(t)\widehat{\gamma}_+(t) + \gamma_+(t)\widehat{\gamma}_-(t)}{\mathrm{e}^{t/2g}-1}\right]\\ = 2\sum_{n\geqslant 1}\left[(2n-1)\gamma_{2n-1}h_{2n-1} - (2n)\gamma_{2n}h_{2n}\right].\end{aligned} \tag{C.8}$$

We note that the expression on the left-hand side of this relation is invariant under exchange $\widehat{\gamma}_\pm(t) \leftrightarrow \gamma_\pm(t)$. Therefore, the right-hand side should also be invariant under (C.5) leading to

$$\widehat{\gamma}_1 = 2\sum_{n\geqslant 1}\left[(2n-1)\gamma_{2n-1}h_{2n-1} - (2n)\gamma_{2n}h_{2n}\right]. \tag{C.9}$$

Replacing h_{2n-1} and h_{2n} by their expressions (C.4) and taking into account (2.10), we obtain that $\widehat{\gamma}_1$ is given by the integral involving the functions $\gamma_\pm(t)$. It takes a much simpler form when expressed in terms of the functions $\Gamma_\pm(t)$ defined in (2.4):

$$\widehat{\gamma}_1 = -\frac{m'}{\pi}\int_0^\infty \mathrm{d}t\left[\frac{\pi g}{t^2+\pi^2 g^2}\left(\Gamma_-(t) - \Gamma_+(t)\right) + \frac{t}{t^2+\pi^2g^2}\left(\Gamma_-(t)+\Gamma_+(t)\right)\right]. \tag{C.10}$$

Making use of identities

$$\begin{aligned}\frac{\pi g}{t^2+\pi^2 g^2} &= \int_0^\infty \mathrm{d}u\,\mathrm{e}^{-\pi g u}\cos(ut),\\ \frac{t}{t^2+\pi^2 g^2} &= \int_0^\infty \mathrm{d}u\,\mathrm{e}^{-\pi g u}\sin(ut),\end{aligned} \tag{C.11}$$

we rewrite $\widehat{\gamma}_1(g)$ as

$$\begin{aligned}\widehat{\gamma}_1 = -\frac{m'}{\pi}\int_0^\infty \mathrm{d}u\,\mathrm{e}^{-\pi g u}\Bigg[&\int_0^\infty \mathrm{d}t\cos(ut)\left(\Gamma_-(t)-\Gamma_+(t)\right)\\ &+\int_0^\infty \mathrm{d}t\sin(ut)\left(\Gamma_-(t)+\Gamma_+(t)\right)\Bigg].\end{aligned} \tag{C.12}$$

Let us spit the u-integral into $0 \leqslant u \leqslant 1$ and $u > 1$. We observe that for $u^2 \leqslant 1$, the t-integrals in this relation are given by (2.6). Then, we perform integration over $u \geqslant 1$ and find after some algebra (with $\Gamma(t) = \Gamma_+(t) + \mathrm{i}\Gamma_-(t)$)

$$\widehat{\gamma}_1 = -\frac{2m'}{\pi^2 g}(1-\mathrm{e}^{-\pi g}) - \frac{\sqrt{2}m'}{\pi}\mathrm{e}^{-\pi g}\mathrm{Re}\left[\int_0^\infty \frac{\mathrm{d}t}{t+\mathrm{i}\pi g}\,\mathrm{e}^{\mathrm{i}(t-\pi/4)}\Gamma(t)\right]. \tag{C.13}$$

Substituting this relation into (C.7), we arrive at (4.36).

Appendix D. Relation to the Whittaker functions

In this appendix, we summarize the properties of special functions that we encountered in our analysis.

D.1. Integral representations

Let us first consider the functions $V_n(x)$ (with $n = 0, 1$) introduced in (2.41). As follows from their integral representation, $V_0(x)$ and $V_1(x)$ are entire functions on a complex x-plane. Changing the integration variable in (2.41) as $u = 2t - 1$ and $u = 1 - 2t$, we obtain two equivalent representations:

$$V_n(x) = \frac{1}{\pi}2^{3/2-n}\,\mathrm{e}^{x}\int_0^1 \mathrm{d}t\,t^{-1/4}(1-t)^{1/4-n}\,\mathrm{e}^{-2tx},$$
$$= \frac{1}{\pi}2^{3/2-n}\,\mathrm{e}^{-x}\int_0^1 \mathrm{d}t\,t^{1/4-n}(1-t)^{-1/4}\,\mathrm{e}^{2tx}, \qquad \text{(D.1)}$$

which give rise to the following expressions for $V_n(x)$ (with $n = 0, 1$) in terms of the Whittaker functions of the first kind:

$$V_n(x) = 2^{-n}\frac{\Gamma\left(\frac{5}{4}-n\right)}{\Gamma\left(\frac{5}{4}\right)\Gamma(2-n)}(2x)^{n/2-1}M_{n/2-1/4,1/2-n/2}(2x),$$
$$= 2^{-n}\frac{\Gamma\left(\frac{5}{4}-n\right)}{\Gamma\left(\frac{5}{4}\right)\Gamma(2-n)}(-2x)^{n/2-1}M_{1/4-n/2,1/2-n/2}(-2x). \qquad \text{(D.2)}$$

In distinction with $V_n(x)$, the Whittaker function $M_{n/2-1/4,1/2-n/2}(2x)$ is an analytical function of x on the complex plane with the cut along the negative semi-axis. The same is true for the factor $(2x)^{n/2-1}$ so that the product of two functions on the right-hand side of (D.2) is a single-valued analytical function in the whole complex plane. The two representations (D.2) are equivalent in virtue of the relation

$$M_{n/2-1/4,1/2-n/2}(2x) = \mathrm{e}^{\pm\mathrm{i}\pi(1-n/2)}M_{1/4-n/2,1/2-n/2}(-2x) \qquad (\text{for}\quad \mathrm{Im}x \gtrless 0), \qquad \text{(D.3)}$$

where the upper and lower signs in the exponent correspond to $\mathrm{Im}x > 0$ and $\mathrm{Im}x < 0$, respectively.

Let us now consider the functions $U_0^{\pm}(x)$ and $U_1^{\pm}(x)$. For real positive x, they have an integral representation (2.41). It is easy to see that four different integrals in (2.41) can be found as special cases of the following generic integral:

$$U_{ab}(x) = \frac{1}{2}\int_1^{\infty}\mathrm{d}u\,\mathrm{e}^{-x(u-1)}(u+1)^{a+b-1/2}(u-1)^{b-a-1/2}, \qquad \text{(D.4)}$$

defined for $x > 0$. Changing the integration variable as $u = t/x + 1$, we obtain

$$U_{ab}(x) = 2^{a+b-3/2}x^{a-b-1/2}\int_0^{\infty}\mathrm{d}t\,\mathrm{e}^{-t}t^{b-a-1/2}\left(1+\frac{t}{2x}\right)^{a+b-1/2}. \qquad \text{(D.5)}$$

The integral entering this relation can be expressed in terms of the Whittaker functions of the second kind or an equivalently confluent hypergeometric function of the second kind:

$$U_{ab}(x) = 2^{b-3/2}\Gamma\left(\frac{1}{2}-a+b\right)x^{-b-1/2}\,\mathrm{e}^{x}\,W_{ab}(2x),$$
$$= \frac{1}{2}\Gamma\left(\frac{1}{2}-a+b\right)\mathrm{U}\left(\frac{1}{2}-a+b, 1+2b; 2x\right). \qquad \text{(D.6)}$$

This relation can be used to analytically continue $U_{ab}(x)$ from $x > 0$ to the whole complex x-plane with the cut along the negative semi-axis. Matching (D.4) to (2.41), we obtain the following relations for the functions $U_0^{\pm}(x)$ and $U_1^{\pm}(x)$:

$$U_0^+(x) = \frac{1}{2}\Gamma\left(\frac{5}{4}\right) x^{-1}\,\mathrm{e}^x\,W_{-1/4,1/2}(2x), \qquad U_1^+(x) = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)(2x)^{-1/2}\,\mathrm{e}^x\,W_{1/4,0}(2x),$$
$$U_0^-(x) = \frac{1}{2}\Gamma\left(\frac{3}{4}\right) x^{-1}\,\mathrm{e}^x\,W_{1/4,1/2}(2x), \qquad U_1^-(x) = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)(2x)^{-1/2}\,\mathrm{e}^x\,W_{-1/4,0}(2x). \tag{D.7}$$

The functions $V_1(\pm x), U_1^{\pm}(x)$ and $V_0(\pm x), U_0^{\pm}(x)$ satisfy the same Whittaker differential equation and, as a consequence, they satisfy Wronskian relations

$$V_1(-x)U_0^-(x) - V_0(-x)U_1^-(x) = V_1(x)U_0^+(x) + V_0(x)U_1^+(x) = \frac{\mathrm{e}^x}{x}. \tag{D.8}$$

The same relations also follow from (B.7) for $x = \pm y$. In addition,

$$U_0^+(x)U_1^-(-x) + U_1^+(x)U_0^-(-x) = \frac{\pi}{2\sqrt{2x}}\,\mathrm{e}^{\pm\frac{3\mathrm{i}\pi}{4}}, \qquad (\text{for}\quad \mathrm{Im}x \gtrless 0). \tag{D.9}$$

Combining together (D.8) and (D.9), we obtain the following relations between the functions:

$$V_0(x) = \frac{2\sqrt{2}}{\pi}\,\mathrm{e}^{\mp\frac{3\mathrm{i}\pi}{4}}\left[\mathrm{e}^x U_0^-(-x) + \mathrm{e}^{-x}U_0^+(x)\right],$$
$$V_1(x) = \frac{2\sqrt{2}}{\pi}\,\mathrm{e}^{\mp\frac{3\mathrm{i}\pi}{4}}\left[\mathrm{e}^x U_1^-(-x) - \mathrm{e}^{-x}U_1^+(x)\right], \tag{D.10}$$

where the upper and lower signs correspond to Im $x > 0$ and Im $x < 0$, respectively.

At first sight, relations (D.10) look surprising since $V_0(x)$ and $V_1(x)$ are entire functions in the complex x-plane, while $U_0^{\pm}(x)$ and $U_1^{\pm}(x)$ are single-valued functions in the same plane but with a cut along the negative semi-axis. Indeed, one can use relations (D.8) and (D.9) to compute the discontinuity of these functions across the cut as

$$\Delta U_0^{\pm}(-x) = \pm\frac{\pi}{4}\,\mathrm{e}^{-x}V_0(\mp x)\theta(x),$$
$$\Delta U_1^{\pm}(-x) = -\frac{\pi}{4}\,\mathrm{e}^{-x}V_1(\mp x)\theta(x), \tag{D.11}$$

where $\Delta U(-x) \equiv \lim_{\epsilon\to 0}[U(-x+\mathrm{i}\epsilon) - U(-x-\mathrm{i}\epsilon)]/(2\mathrm{i})$ and $\theta(x)$ is a step function. Then, one verifies with the help of these identities that the linear combinations of U-functions on the right-hand side of (D.10) have zero discontinuity across the cut and, therefore, they are well defined in the whole complex plane.

D.2. Asymptotic expansions

For our purposes, we need an asymptotic expansion of functions $V_n(x)$ and $U_n^{\pm}(x)$ at large real x. Let us start with the latter functions and consider a generic integral (D.6).

To find an asymptotic expansion of the function $U_{ab}(x)$ at large x, it suffices to replace the last factor in the integrand (D.6) in powers of $t/(2x)$ and integrate term by term. In this way, we find from (D.6) and (D.7)

$$U_0^+(x) = (2x)^{-5/4}\Gamma\left(\frac{5}{4}\right)F\left(\frac{1}{4},\frac{5}{4}|-\frac{1}{2x}\right) = (2x)^{-5/4}\Gamma\left(\frac{5}{4}\right)\left[1 - \frac{5}{32x} + \cdots\right],$$
$$U_0^-(x) = (2x)^{-3/4}\Gamma\left(\frac{3}{4}\right)F\left(-\frac{1}{4},\frac{3}{4}|-\frac{1}{2x}\right) = (2x)^{-3/4}\Gamma\left(\frac{3}{4}\right)\left[1 + \frac{3}{32x} + \cdots\right],$$

$$U_1^+(x) = (2x)^{-1/4}\frac{1}{2}\Gamma\left(\frac{1}{4}\right)F\left(\frac{1}{4},\frac{1}{4}|-\frac{1}{2x}\right) = (2x)^{-1/4}\frac{1}{2}\Gamma\left(\frac{1}{4}\right)\left[1-\frac{1}{32x}+\cdots\right],$$
$$U_1^-(x) = (2x)^{-3/4}\frac{1}{2}\Gamma\left(\frac{3}{4}\right)F\left(\frac{3}{4},\frac{3}{4}|-\frac{1}{2x}\right) = (2x)^{-3/4}\frac{1}{2}\Gamma\left(\frac{3}{4}\right)\left[1-\frac{9}{32x}+\cdots\right], \tag{D.12}$$

where the function $F(a,b|-\frac{1}{2x})$ is defined in (2.56).

Note that the expansion coefficients in (D.12) grow factorially to higher orders but the series are Borel summable for $x > 0$. For $x < 0$, one has to distinguish the functions $U_n^{\pm}(x+\mathrm{i}\epsilon)$ and $U_n^{\pm}(x-\mathrm{i}\epsilon)$ (with $\epsilon \to 0$) which define the analytical continuation of the function $U_n^{\pm}(x)$ to the upper and lower edges of the cut, respectively. In contrast with this, the functions $V_n(x)$ are well defined on the whole real axis. Still, to make use of relations (D.10) we have to specify the U-functions on the cut. As an example, let us consider $V_0(-\pi g)$ in the limit $g \to \infty$ and apply (D.10)

$$V_0(-\pi g) = \frac{2\sqrt{2}}{\pi}\,\mathrm{e}^{-\frac{3i\pi}{4}}\,\mathrm{e}^{\pi g}\left[U_0^+(-\pi g+\mathrm{i}\epsilon)+\mathrm{e}^{-2\pi g}U_0^-(\pi g)\right], \tag{D.13}$$

where $\epsilon \to 0$ and we have chosen to define the U-functions on the upper edge of the cut. Written in this form, both terms inside the square brackets are well defined separately. Replacing $U_0^{\pm}$ functions in (D.13) by their expressions (D.12) in terms of F-functions, we find

$$V_0(-\pi g) = \frac{(2\pi g)^{-5/4}\,\mathrm{e}^{\pi g}}{\Gamma\left(\frac{3}{4}\right)}\left[F\left(\frac{1}{4},\frac{5}{4}|\frac{1}{2\pi g}+\mathrm{i}\epsilon\right)+\Lambda^2 F\left(-\frac{1}{4},\frac{3}{4}|-\frac{1}{2\pi g}\right)\right], \tag{D.14}$$

with Λ^2 being given by

$$\Lambda^2 = \sigma\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)}\,\mathrm{e}^{-2\pi g}(2\pi g)^{1/2}, \qquad \sigma = \mathrm{e}^{-\frac{3i\pi}{4}}. \tag{D.15}$$

Since the second term on the right-hand side of (D.14) is exponentially suppressed at large g, we may treat it as a nonperturbative correction. Repeating the same analysis for $V_1(-\pi g)$, we obtain from (D.10) and (D.12)

$$V_1(-\pi g) = \frac{(2\pi g)^{-5/4}\,\mathrm{e}^{\pi g}}{2\Gamma\left(\frac{3}{4}\right)}\left[8\pi g F\left(\frac{1}{4},\frac{1}{4}|\frac{1}{2\pi g}+\mathrm{i}\epsilon\right)+\Lambda^2 F\left(\frac{3}{4},\frac{3}{4}|-\frac{1}{2\pi g}\right)\right]. \tag{D.16}$$

We would like to stress that the '$+\mathrm{i}\epsilon$' prescription in the first term in (D.14) and the phase factor $\sigma = \mathrm{e}^{-\frac{3i\pi}{4}}$ in (D.15) follow unambiguously from (D.13). Had we defined the U-functions on the lower edge of the cut, we would get the expression for $V_0(-\pi g)$ with the '$-\mathrm{i}\epsilon$' prescription and the phase factor $\mathrm{e}^{\frac{3i\pi}{4}}$. The two expressions are however equivalent since discontinuity of the first term in (D.14) compensates the change of the phase factor in front of the second term:

$$F\left(\frac{1}{4},\frac{5}{4}|\frac{1}{2\pi g}+\mathrm{i}\epsilon\right)-F\left(\frac{1}{4},\frac{5}{4}|\frac{1}{2\pi g}-\mathrm{i}\epsilon\right) = \frac{\mathrm{i}\sqrt{2}\Lambda^2}{\sigma}F\left(-\frac{1}{4},\frac{3}{4}|-\frac{1}{2\pi g}\right). \tag{D.17}$$

If one neglected the '$+\mathrm{i}\epsilon$' prescription in (D.13) and formally expanded the first term in (D.14) in powers of $1/g$, this would lead to non-Borel summable series. This series suffers from Borel ambiguity which are exponentially small for large g and produce the contribution of the same order as the second term on the right-hand side of (D.14). Relation (D.14) suggests how to give a meaning to this series. Namely, one should first resum the series for negative g where it is Borel summable and, then, analytically continue it to the upper edge of the cut at positive g.

Appendix E. Expression for the mass gap

In this appendix, we derive the expression for the mass gap (4.4). To this end, we replace $\Gamma(4\pi git)$ in (4.1) by its expression (4.2) and perform integration over t on the right-hand side of (4.1). We recall that, in relation (4.2), $V_{0,1}(4\pi gt)$ are entire functions of t, while $f_{0,1}(4\pi gt)$ are meromorphic functions defined in (4.3). It is convenient to decompose $\Gamma(4\pi git)/\left(t+\frac{1}{4}\right)$ into a sum of simple poles as

$$\frac{\Gamma(4\pi git)}{t+\frac{1}{4}} = \sum_{k=0,1} f_k(-\pi g)\frac{V_k(4\pi gt)}{t+\frac{1}{4}} + \sum_{k=0,1}\frac{f_k(4\pi gt)-f_k(-\pi g)}{t+\frac{1}{4}}V_k(4\pi gt), \tag{E.1}$$

where the second term is regular at $t=-1/4$. Substituting this relation into (4.1) and replacing $f_k(4\pi gt)$ by their expressions (4.3), we encounter the following integral:

$$R_k(4\pi gs) = \mathrm{Re}\left[\int_0^{-\mathrm{i}\infty} \mathrm{d}t\, \mathrm{e}^{-4\pi gt-\mathrm{i}\pi/4}\frac{V_k(4\pi gt)}{t-s}\right] = \mathrm{Re}\left[\int_0^{-\mathrm{i}\infty} \mathrm{d}t\, \mathrm{e}^{-t-\mathrm{i}\pi/4}\frac{V_k(t)}{t-4\pi gs}\right]. \tag{E.2}$$

Then, the integral in (4.1) can be expressed in terms of the R-function as

$$\begin{aligned}\mathrm{Re}\left[\int_0^{-\mathrm{i}\infty}\mathrm{d}t\,\mathrm{e}^{-4\pi gt-\mathrm{i}\pi/4}\frac{\Gamma(4\pi git)}{t+\frac{1}{4}}\right] &= f_0(-\pi g)R_0(-\pi g) + f_1(-\pi g)R_1(-\pi g)\\ &\quad - \sum_{n\geqslant 1}\frac{nc_+(n,g)}{n+\frac{1}{4}}\left[U_1^+(4\pi ng)R_0(4\pi gn) + U_0^+(4\pi ng)R_1(4\pi gn)\right]\\ &\quad + \sum_{n\geqslant 1}\frac{nc_-(n,g)}{n-\frac{1}{4}}[U_1^-(4\pi ng)R_0(-4\pi gn) - U_0^-(4\pi ng)R_1(-4\pi gn)],\end{aligned} \tag{E.3}$$

where the last two lines correspond to the second sum on the right-hand side of (E.1), and we took into account the fact that the coefficients $c_\pm(n,g)$ are real.

Let us evaluate the integral (E.2) and choose for simplicity $R_0(s)$. We have to distinguish two cases: $s>0$ and $s<0$. For $s>0$, we have

$$\begin{aligned}R_0(s) &= -\mathrm{Re}\left[\mathrm{e}^{-\mathrm{i}\pi/4}\int_{-\infty}^1 \mathrm{d}v\,\mathrm{e}^{-(1-v)s}\int_0^{-\mathrm{i}\infty}\mathrm{d}t\,\mathrm{e}^{-vt}V_0(t)\right]\\ &= \frac{\sqrt{2}}{\pi}\mathrm{Re}\left[\mathrm{e}^{-\mathrm{i}\pi/4}\int_{-\infty}^1\mathrm{d}v\,\mathrm{e}^{-(1-v)s}\int_{-1}^1\mathrm{d}u\frac{(1+u)^{1/4}(1-u)^{-1/4}}{u-v-\mathrm{i}\epsilon}\right],\end{aligned} \tag{E.4}$$

where in the second relation we replaced $V_0(t)$ by its integral representation (2.41). Integration over u can be carried out with the help of identity

$$\frac{1}{\sqrt{2}\pi}\int_{-1}^1\mathrm{d}u\frac{(1+u)^{1/4-k}(1-u)^{-1/4}}{u-v-\mathrm{i}\epsilon} = \delta_{k,0} - (v+1)^{-k}\times\begin{cases}\left(\frac{v+1}{v-1}\right)^{1/4}, & v^2>1\\ \mathrm{e}^{-\mathrm{i}\pi/4}\left(\frac{1+v}{1-v}\right)^{1/4}, & v^2<1.\end{cases} \tag{E.5}$$

In this way, we obtain from (E.4)

$$R_0(s) \stackrel{s>0}{=} \sqrt{2}\left[\frac{1}{s} - \int_{-\infty}^{-1}\mathrm{d}v\,\mathrm{e}^{-(1-v)s}\left(\frac{v+1}{v-1}\right)^{1/4}\right] = \sqrt{2}\left[\frac{1}{s} - 2\,\mathrm{e}^{-2s}U_0^+(s)\right], \tag{E.6}$$

with the function $U_0^+(s)$ defined in (2.41). In a similar manner, for $s<0$ we get

$$R_0(s) \stackrel{s\leqslant 0}{=} \sqrt{2}\left[\frac{1}{s} + 2U_0^-(-s)\right], \tag{E.7}$$

together with

$$R_1(s) = 2\sqrt{2}\left[\theta(-s)U_1^-(-s) + \theta(s)\,\mathrm{e}^{-2s}U_1^+(s)\right]. \tag{E.8}$$

Then, we substitute relations (E.6))–((E.8) into (E.3) and find

$$\begin{aligned}\mathrm{Re}&\left[\int_0^{-\mathrm{i}\infty} \mathrm{d}t\ \mathrm{e}^{-4\pi g t - \mathrm{i}\pi/4}\frac{\Gamma(4\pi g \mathrm{i}t)}{t+\frac{1}{4}}\right]\\ &= 2\sqrt{2}f_0(-\pi g)\left[U_0^-(\pi g) - \frac{1}{2\pi g}\right]\\ &\quad + 2\sqrt{2}f_1(-\pi g)U_1^-(\pi g) + \frac{\sqrt{2}}{\pi g}\left[f_0(-\pi g)+1\right],\end{aligned} \tag{E.9}$$

where the last term on the right-hand side corresponds to the last two lines in (E.3) (see equation (2.40)). Substitution of (E.9) into (4.1) yields the expression for the mass scale (4.4).

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